An approximate spectral representation and explicit bounds for Green functions of Fuchsian groups

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Abstract. We study the Green function gr_Γ for the Laplace operator on the quotient of the hyperbolic plane by a cofinite Fuchsian group Γ. We use a limiting procedure, starting from the resolvent kernel, and lattice point estimates for the action of Γ on the hyperbolic plane to prove an "approximate spectral representation" for gr_Γ. Combining this with bounds on Maaß forms and Eisenstein series for Γ, we prove explicit bounds on gr_Γ.

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1. Introduction and statement of results

The hyperbolic plane $\mathbf{H}$ is the unique two-dimensional, complete, connected and simply connected Riemannian manifold with constant Gaussian curvature -1. We identify $\mathbf{H}$ with the complex upper half-plane; this gives $\mathbf{H}$ a complex structure. In terms of the standard coordinate $z = x + iy$, the Riemannian metric is

$$\frac{dz\,d\bar{z}}{(\Im z)^2} = \frac{dx^2 + dy^2}{y^2},$$

and the associated volume form is

$$\mu_{\mathbf{H}} = \frac{i\,dz \wedge d\bar{z}}{2(\Im z)^2} = \frac{dx \wedge dy}{y^2}.$$

Instead of using the geodesic distance $r(z, w)$ on $\mathbf{H}$ directly, we use the more convenient function

$$\begin{aligned} u(z, w) &= \cosh r(z, w) \\ &= 1 + \frac{|z - w|^2}{2(\Im z)(\Im w)}. \end{aligned}$$

Let Δ denote the Laplace–Beltrami operator on $\mathbf{H}$, given by

$$\Delta = y^2(\partial_x^2 + \partial_y^2).$$

The Green function for Δ is the unique smooth real-valued function $\mathrm{gr}_{\mathbf{H}}$ outside the diagonal on $\mathbf{H} \times \mathbf{H}$ satisfying

$$\begin{gathered} \mathrm{gr}_{\mathbf{H}}(z, w) = \frac{1}{2\pi} \log|z - w| + O(1) \quad \text{as } z \to w, \\ \Delta\,\mathrm{gr}_{\mathbf{H}}(\ \ , w) = \delta_w \quad \text{for all } w \in \mathbf{H}, \\ \mathrm{gr}_{\mathbf{H}}(z, w) = O(u(z, w)^{-1}) \quad \text{as } u(z, w) \to \infty, \end{gathered}$$

where Δ is taken with respect to the first variable. It is given by

$$\mathrm{gr}_{\mathbf{H}}(z, w) = -L(u(z, w)),$$

where

$$L(u) = \frac{1}{4\pi} \log \frac{u + 1}{u - 1}. \tag{1.1}$$

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The group $\mathrm{SL}_2(\mathbf{R})$ acts on $\mathbf{H}$ by isometries. Under the identification of $\mathbf{H}$ with the complex upper half-plane, this action on $\mathbf{H}$ is the restriction of the action on $\mathbf{P}^1(\mathbf{C})$ by Möbius transformations. Elements of $\mathrm{SL}_2(\mathbf{R}) \setminus \{\pm 1\}$ are classified according to their fixed points in $\mathbf{P}^1(\mathbf{C})$ as elliptic (two conjugate fixed points in $\mathbf{P}^1(\mathbf{C}) \setminus \mathbf{P}^1(\mathbf{R})$), parabolic (a unique fixed point in $\mathbf{P}^1(\mathbf{R})$), and hyperbolic (two distinct fixed points in $\mathbf{P}^1(\mathbf{R})$). This terminology also applies to conjugacy classes.

A *Fuchsian group* is a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$. A *cofinite Fuchsian group* is a Fuchsian group Γ such that the volume of $\Gamma\backslash\mathbf{H}$ with respect to the measure induced by $\mu_{\mathbf{H}}$ is finite. We will exclusively consider cofinite Fuchsian groups, and for such a group Γ we write

$$\mathrm{vol}_\Gamma = \int_{\Gamma\backslash\mathbf{H}} \mu_{\mathbf{H}}.$$

Remark. We define integration on $\Gamma\backslash\mathbf{H}$ in a stack-like way, so the above integral is $1/\#(\Gamma \cap \{\pm 1\})$ times the integral over $\Gamma\backslash\mathbf{H}$ viewed as a Riemann surface. This implies that if f is a Γ-invariant function on $\mathbf{H}$ and Γ' is a subgroup of finite index in Γ, then

$$\int_{\Gamma'\backslash\mathbf{H}} f\mu_{\mathbf{H}} = (\Gamma : \Gamma') \cdot \int_{\Gamma\backslash\mathbf{H}} f\mu_{\mathbf{H}}.$$

Furthermore, this definition justifies the method of "unfolding": if f is a smooth function with compact support on $\mathbf{H}$ and F is the function on $\Gamma\backslash\mathbf{H}$ defined by

$$F(z) = \sum_{\gamma\in\Gamma} f(\gamma z),$$

then

$$\int_{\Gamma\backslash\mathbf{H}} F\mu_{\mathbf{H}} = \int_{\mathbf{H}} f\mu_{\mathbf{H}}.$$

Let Γ be a cofinite Fuchsian group. The restriction of the Laplace operator Δ to the space of smooth and bounded Γ-invariant functions on $\mathbf{H}$ can be extended to an (unbounded, densely defined) self-adjoint operator on the Hilbert space $\mathrm{L}^2(\Gamma\backslash\mathbf{H})$, which we denote by Δ_Γ.

The operator Δ_Γ is invertible on the orthogonal complement of the constant functions in the following sense: there exists a unique bounded self-adjoint operator G_Γ on $\mathrm{L}^2(\Gamma\backslash\mathbf{H}, \mu_{\mathbf{H}})$ such that for all smooth and bounded functions f on $\Gamma\backslash\mathbf{H}$ the function $G_\Gamma f$ satisfies

$$\Delta_\Gamma G_\Gamma f = f - \frac{1}{\mathrm{vol}_\Gamma}\int_{\Gamma\backslash\mathbf{H}} f\mu_{\mathbf{H}} \quad \text{and} \quad \int_{\Gamma\backslash\mathbf{H}} G_\Gamma f \mu_{\mathbf{H}} = 0.$$

There exists a unique function gr_Γ on $\mathbf{H}\times\mathbf{H}$ that is Γ-invariant in both variables separately, satisfies $\mathrm{gr}_\Gamma(z,w) = \mathrm{gr}_\Gamma(w,z)$, is smooth except for logarithmic singularities at points of the form $(z, \gamma z)$, and has the property that if f is a smooth and bounded Γ-invariant function on $\mathbf{H}$, then the function $G_\Gamma f$ is given by

$$G_\Gamma f(z) = \int_{w\in\Gamma\backslash\mathbf{H}} \mathrm{gr}_\Gamma(z,w) f(w) \mu_{\mathbf{H}}(w).$$

The function gr_Γ is called the *Green function of the Fuchsian group* Γ.

In this paper, we study gr_Γ quantitatively, with the goal of obtaining explicit upper and lower bounds. One result that can be stated without introducing too much notation is the following.

Theorem 1.1 (corollary of Theorem 4.1). *Let Γ_0 be a cofinite Fuchsian group, let Y_0 be a compact subset of $\Gamma_0\backslash\mathbf{H}$, and let $\delta > 1$ and $\eta > 0$ be real numbers. There exist real numbers A and B such that the following holds. Let Γ be a subgroup of finite index in Γ_0 such that all non-zero eigenvalues of $-\Delta_\Gamma$ are at least η. Then for all $z, w \in \mathbf{H}$ whose images in $\Gamma_0\backslash\mathbf{H}$ lie in Y_0, we have*

$$A \le \mathrm{gr}_\Gamma(z,w) + \sum_{\substack{\gamma\in\Gamma \\ u(z,\gamma w)\le\delta}} \bigl(L(u(z,\gamma w)) - L(\delta)\bigr) \le B.$$

Let us give a very brief overview of the article. In Section 2, we collect known results about Fuchsian groups. Most importantly, we make use of the hyperbolic lattice point problem and the techniques used to attack this problem. In Section 3, we use these results, together with a construction of the Green function involving the resolvent kernel, to "sandwich" the Green function gr_Γ (with the logarithmic singularity removed) between two functions that, unlike gr_Γ itself, admit spectral representations. In Section 4, we bound these functions in a way that lends itself to explicit evaluation. As an example, we find explicit constants A and B as in Theorem 1.1 in the case where $\Gamma_0 = \mathrm{SL}_2(\mathbf{Z})$, $\Gamma \subseteq \Gamma_0$ is a congruence subgroup, $\delta = 2$ and Y_0 is the compact subset of $\Gamma_0\backslash\mathbf{H}$ corresponding to the points z in the standard fundamental domain of $\mathrm{SL}_2(\mathbf{Z})$ such that $\Im z \le 2$. In Section 5, we use the bounds given by Theorem 1.1 to deduce bounds on $\mathrm{gr}_\Gamma(z,w)$ in the case where Y is obtained by cutting out discs around the cusps of Γ and where z, w or both are in such a disc. Finally, a number of bounds on Legendre functions that we will need have been collected in an appendix.

In a forthcoming paper, we will use the results in this article to obtain explicit bounds on the *canonical Green function* of a modular curve X. This function is defined similarly to $\mathrm{gr}_{\mathbf{H}}$, using the *canonical* $(1,1)$*-form* on X instead of $\mu_{\mathbf{H}}$. It plays a fundamental role in Arakelov theory; see Arakelov [1] and Faltings [6]. Explicit bounds on canonical Green functions of modular curves are relevant to the work of Edixhoven, Couveignes et al. [3] and the author [2] on computing two-dimensional representations of the absolute Galois group of $\mathbf{Q}$ that are associated to Hecke eigenforms over finite fields.

This article may be compared with earlier work of Jorgenson and Kramer on bounding canonical and hyperbolic Green functions of compact Riemann surfaces [11, especially Theorem 4.5]. Jorgenson and Kramer consider compact Riemann surfaces X of genus at least 2, which can be obtained as $X = \Gamma\backslash\mathbf{H}$ for a cofinite Fuchsian group Γ without elliptic and parabolic elements. They obtain bounds on the hyperbolic Green function by comparing it to the heat kernel on X. Our method, too, starts with comparing the Green function with a kernel that can be obtained as a sum over elements of Γ, but the subsequent arguments are rather different. Let us note some of the differences. First, we allow arbitrary cofinite Fuchsian groups, which is the natural setting for modular curves. Second, the procedure that we apply in § 3.1 to construct the Green function as a limit of a family of kernels K_a for $a \to 1$ leads to bounds that are independent of the specific family. We take K_a to be the resolvent kernel with parameter $a \to 1$, but the heat kernel with parameter $t \to \infty$ could have been used with the same result; see [2, § II.5.2]. Finally, our bounds are much easier to make explicit than those in [11]; this is illustrated in § 4.3.

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2. Tools

2.1. Cusps

Let Γ be a cofinite Fuchsian group. The cusps of Γ correspond to the conjugacy classes of non-trivial maximal parabolic subgroups in Γ. Every such subgroup has a unique fixed point in $\mathbf{P}^1(\mathbf{R})$. For every cusp $\mathfrak{c}$ we choose a representative of the corresponding conjugacy class and denote it by $\Gamma_\mathfrak{c}$.

Let $\mathfrak{c}$ be a cusp of Γ. We fix an element $\sigma_\mathfrak{c} \in \mathrm{SL}_2(\mathbf{R})$ such that $\sigma_\mathfrak{c}\infty \in \mathbf{P}^1(\mathbf{R})$ is the unique fixed point of $\Gamma_\mathfrak{c}$ in $\mathbf{P}^1(\mathbf{R})$ and such that

$$\{\pm 1\}\sigma_\mathfrak{c}^{-1}\Gamma_\mathfrak{c}\sigma_\mathfrak{c} = \{\pm 1\}\bigl\{\bigl(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\bigr) \bigm| b \in \mathbf{Z}\bigr\}.$$

Such a $\sigma_\mathfrak{c}$ exists and is unique up to multiplication from the right by a matrix of the form $\pm\bigl(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\bigr)$ with $x \in \mathbf{R}$; see Iwaniec [10, § 2.2]. We define

$$\begin{aligned} q_\mathfrak{c}\colon \mathbf{H} &\to \mathbf{C} \\ z &\mapsto \exp(2\pi i \sigma_\mathfrak{c}^{-1} z) \end{aligned}$$

and

$$\begin{aligned} y_{\mathfrak{c}}\colon \mathbf{H} &\to (0,\infty) \\ z &\mapsto \Im\sigma_{\mathfrak{c}}^{-1}z = -\frac{\log|q_{\mathfrak{c}}(z)|}{2\pi}. \end{aligned}$$

For all $\gamma \in \Gamma$, we write

$$C_{\mathfrak{c}}(\gamma) = |c| \quad \text{if } \sigma_{\mathfrak{c}}^{-1}\gamma\sigma_{\mathfrak{c}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we have

$$\Gamma_{\mathfrak{c}} = \{\gamma \in \Gamma \mid C_{\mathfrak{c}}(\gamma) = 0\}.$$

It is known that the set $\{C_{\mathfrak{c}}(\gamma) \mid \gamma \in \Gamma, \gamma \notin \Gamma_{\mathfrak{c}}\}$ is bounded from below by a positive number, and that if ϵ is a real number satisfying the inequality

$$0 < \epsilon \le \min_{\substack{\gamma\in\Gamma \\ \gamma\notin\Gamma_{\mathfrak{c}}}} C_{\mathfrak{c}}(\gamma), \tag{2.1}$$

then for all $z \in \mathbf{H}$ and $\gamma \in \Gamma$ one has the implication

$$y_{\mathfrak{c}}(z) > 1/\epsilon \text{ and } y_{\mathfrak{c}}(\gamma z) > 1/\epsilon \implies \gamma \in \Gamma_{\mathfrak{c}}.$$

For any ϵ satisfying (2.1), the image of the strip

$$\{x + iy \mid 0 \le x < 1 \text{ and } y > 1/\epsilon\} \subset \mathbf{H}$$

under the map

$$\mathbf{H} \xrightarrow{\sigma_{\mathfrak{c}}} \mathbf{H} \longrightarrow \Gamma\backslash\mathbf{H}$$

is an open disc $D_{\mathfrak{c}}(\epsilon)$ around $\mathfrak{c}$, and the map $q_{\mathfrak{c}}$ induces a chart on $\Gamma\backslash\mathbf{H}$ identifying $D_{\mathfrak{c}}(\epsilon)$ with the punctured disc $\{z \in \mathbf{C} \mid 0 < |z| < \exp(-2\pi/\epsilon)\}$. A compactification of $\Gamma\backslash\mathbf{H}$ can be obtained by adding a point for every cusp $\mathfrak{c}$ in such a way that $q_{\mathfrak{c}}$ extends to a chart with image equal to the disc $\{z \in \mathbf{C} \mid |z| < \exp(-2\pi/\epsilon)\}$. Let $\bar{D}_{\mathfrak{c}}(\epsilon)$ denote the compactification of $D_{\mathfrak{c}}(\epsilon)$ obtained by adding the boundary $\partial\bar{D}_{\mathfrak{c}}(\epsilon)$ in $\Gamma\backslash\mathbf{H}$ and the cusp $\mathfrak{c}$.

Remark. Let us fix a point $w \in \mathbf{H}$ and write Γ_w for the stabilisator of w in Γ. The behaviour of $\mathrm{gr}_\Gamma(z,w)$ as $z \to w$ is

$$\mathrm{gr}_\Gamma(z,w) = \frac{\#\Gamma_w}{2\pi}\log|z-w| \text{ as } z \to w.$$

Furthermore, the behaviour of $\mathrm{gr}_\Gamma(z,w)$ as z moves toward a cusp $\mathfrak{c}$ of Γ is

$$\mathrm{gr}_\Gamma(z,w) = \frac{1}{\mathrm{vol}_\Gamma}\log y_{\mathfrak{c}}(z) + O(1) \text{ as } y_{\mathfrak{c}}(z) \to \infty.$$

2.2. The Selberg–Harish-Chandra transform

Let $g\colon [1,\infty) \to \mathbf{R}$ be a smooth function with compact support. The *invariant integral operator attached to* g is the operator T_g defined on smooth functions $f\colon \mathbf{H} \to \mathbf{C}$ by

$$(T_g f)(z) = \int_{w\in\mathbf{H}} g(u(z,w))f(w)\mu_{\mathbf{H}}(w).$$

The Laplace operator Δ commutes with all such operators T_g; see Selberg [16, pages 51–52] or Iwaniec [10, Theorem 1.9]. In fact, every eigenfunction of Δ is also an eigenfunction of all invariant integral operators, and conversely; see Selberg [16, page 55] or Iwaniec [10, Theorems 1.14 and 1.15]. The relation between the eigenvalues of Δ and those of T_g is given by the *Selberg–Harish-Chandra transform* of g. This is a holomorphic function h defined by the following property. Let $f\colon \mathbf{H} \to \mathbf{C}$

be an eigenfunction of $-\Delta$ with eigenvalue $\lambda = s(1-s)$. Then f is also an eigenfunction of T_g, and the eigenvalue depends only on λ; we can therefore define $h(s)$ uniquely such that

$$-\Delta f = s(1-s)f \implies T_g f = h(s)f. \tag{2.2}$$

In particular, taking $f = 1$, we see that

$$h(0) = h(1) = 2\pi \int_1^\infty g(u)du. \tag{2.3}$$

The Selberg–Harish-Chandra transform can be identified with the classical *Mehler–Fock transform*, defined as follows (see Iwaniec [10, equation 1.62′]):

$$h(s) = 2\pi \int_1^\infty g(u)P_{s-1}(u)du. \tag{2.4}$$

Here P_ν is the Legendre function of the first kind of degree ν; see Iwaniec [10, equation 1.43] or any book on special functions, such as Erdélyi et al. [4, § 3.6.1]. The function g can be recovered from h, and we call g the *inverse Selberg–Harish-Chandra transform* of h.

The identity (2.2) holds more generally than just for smooth functions g with compact support; see Selberg [16, pages 60–61]. It will be enough for us to state a slightly weaker, but more convenient sufficient condition (cf. Selberg [16, page 72] or Iwaniec [10, equation 1.63]). Let $\epsilon > 0$ and $\beta > 1$, and let h be a holomorphic function on the strip $\{s \in \mathbf{C} \mid -\epsilon < \Re s < 1+\epsilon\}$ such that $h(s) = h(1-s)$ and such that $s \mapsto |h(s)||s(1-s)|^\beta$ is bounded on this strip. Then the inverse Selberg–Harish-Chandra transform g of h exists, and (2.2) holds for the pair (g, h).

2.3. Spectral theory of the Laplace operator for Fuchsian groups

Let Γ be a cofinite Fuchsian group. The spectrum of $-\Delta_\Gamma$ on $\mathrm{L}^2(\Gamma\backslash\mathbf{H})$ consists of a discrete part and a continuous part.

The discrete spectrum consists of eigenvalues of $-\Delta_\Gamma$ and is of the form $\{\lambda_j\}_{j=0}^\infty$ with

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots, \quad \lambda_j \to \infty \text{ as } j \to \infty.$$

Let $\{\phi_j\}_{j=0}^\infty$ be a corresponding set of eigenfunctions; these are called *automorphic forms of Maaß (of weight 0)*. We may and do assume that they are orthonormal with respect to the inner product on $\mathrm{L}^2(\Gamma\backslash\mathbf{H})$. For each $j \ge 0$, we define $s_j \in \mathbf{C}$ by

$$\lambda_j = s_j(1-s_j),$$

with $s_j \in [1/2, 1]$ if $\lambda_j \le 1/4$. For $\lambda_j > 1/4$, the s_j are only determined up to $s_j \leftrightarrow 1 - s_j$.

The continuous part of the spectrum of $-\Delta_\Gamma$ is the interval $[1/4, \infty)$ with multiplicity equal to the number of cusps of Γ. In particular, the continuous spectrum is absent if Γ has no cusps. The continuous spectrum does not consist of eigenvalues, but corresponds to "wave packets" that can be constructed from *non-holomorphic Eisenstein series* or *Eisenstein–Maaß series*, introduced by Maaß in [13]. These series are defined as follows: for every cusp $\mathfrak{c}$ of Γ the series

$$E_\mathfrak{c}(z, s) = \sum_{\gamma \in \Gamma_\mathfrak{c}\backslash\Gamma} (y_\mathfrak{c}(\gamma z))^s \quad (z \in \mathbf{H}, s \in \mathbf{C} \text{ with } \Re s > 1)$$

converges uniformly on sets of the form $K \times \{s \in \mathbf{C} \mid \Re s \ge \delta\}$ with K a compact subset of $\mathbf{H}$ and $\delta > 1$. In particular, $E_\mathfrak{c}(z, s)$ is a holomorphic function of s.

A crucial ingredient in the spectral theory of automorphic forms is the *meromorphic continuation of Eisenstein series*, due to Selberg [17]. For proofs of this meromorphic continuation and of the other properties of the Eisenstein series that we use, we refer to Hejhal [8, Chapter VI, § 11]. Different constructions of the meromorphic continuation can be found in Faddeev [5, § 4], Hejhal [8, Appendix F] or Iwaniec [10, Chapter 6].

The meromorphic continuation of the Eisenstein series takes the following form. The functions $E_\mathfrak{c}(z, s)$ can be continued to functions of the form $E_\mathfrak{c}(z, s) = H(z, s)/G(s)$, where H is a smooth

function on $\Gamma\backslash\mathbf{H}\times\mathbf{C}$ and both G and H are entire functions of s. These meromorphic continuations have a finite number of simple poles on the segment $(1/2, 1]$ and no other poles in $\{s \in \mathbf{C} \mid \Re s \geq 1/2\}$, and satisfy a functional equation, which we will not write down. For all $s \in \mathbf{C}$ that is not a pole, the function $z \mapsto E_\mathfrak{c}(z, s)$ satisfies the differential equation

$$-\Delta_\Gamma E_\mathfrak{c}(\ \ , s) = s(1-s)E_\mathfrak{c}(\ \ , s).$$

For $s \in \mathbf{C}$ with $\Re s = 1/2$, the Eisenstein–Maaß series $E_\mathfrak{c}(\ \ , s)$ are integrable, but not square-integrable, as functions on $\Gamma\backslash\mathbf{H}$. In contrast, the "wave packets" mentioned above are square-integrable. They figure in Theorem 2.1 below, which is a fundamental result in the theory of automorphic forms.

In the following theorem, and in the rest of the article, we will consider integrals over the line $\Re s = 1/2$. For this we need an orientation on this line; we fix one by requiring that the map $t \mapsto 1/2 + it$ from $\mathbf{R}$ with the usual orientation to the line $\Re s = 1/2$ preserves orientations.

Theorem 2.1 (see Iwaniec [10, Theorems 4.7 and 7.3]; cf. Faddeev [5, Theorem 4.1]). *Every smooth and bounded Γ-invariant function $f\colon \mathbf{H} \to \mathbf{C}$ has the spectral representation*

$$f(z) = \sum_{j=0}^{\infty} b_j\phi_j(z) + \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s=1/2} b_\mathfrak{c}(s)\, E_\mathfrak{c}(z, s)ds, \tag{2.5}$$

where $\mathfrak{c}$ runs over the cusps of Γ and the coefficients b_j and $b_\mathfrak{c}(s)$ are given by

$$b_j = \int_{\Gamma\backslash\mathbf{H}} f\bar{\phi}_j\mu_\mathbf{H} \quad \text{and} \quad b_\mathfrak{c}(s) = \int_{\Gamma\backslash\mathbf{H}} f\bar{E}_\mathfrak{c}(\ \ , s)\mu_\mathbf{H}.$$

The right-hand side of (2.5) *converges to f in the Hilbert space $\mathrm{L}^2(\Gamma\backslash\mathbf{H})$. If in addition the smooth Γ-invariant function $\Delta f\colon \mathbf{H} \to \mathbf{C}$ is bounded, the convergence is uniform on compact subsets of $\mathbf{H}$.*

With regard to the spectral representations provided by this theorem, the effect of the operator G_Γ from the introduction is as follows: if f has the spectral representation (2.5), then $G_\Gamma f$ has the corresponding spectral representation

$$G_\Gamma f(z) = -\sum_{j=1}^{\infty} \frac{b_j}{\lambda_j}\phi_j(z) - \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s=1/2} \frac{b_\mathfrak{c}(s)}{s(1-s)} E_\mathfrak{c}(z, s)ds. \tag{2.6}$$

(Note the absence of the eigenvalue $\lambda_0 = 0$.)

There is an analogous result (Theorem 2.2 below) for functions on $\mathbf{H} \times \mathbf{H}$ that are of the form $\sum_{\gamma\in\Gamma} g(u(z, \gamma w))$, where $g\colon [1, \infty) \to \mathbf{R}$ is the inverse Selberg–Harish-Chandra transform of a function h as at the end of § 2.2. The result involves a type of convergence that we now explain. Let A be a filtered set, and let $\{K_a\}_{a\in A}$ be a family of continuous functions on $\Gamma\backslash\mathbf{H} \times \Gamma\backslash\mathbf{H}$ that are square-integrable in the second variable. If K is a function such that for all compact subsets C of $\Gamma\backslash\mathbf{H}$ we have

$$\lim_{a\in A}\left(\sup_{z,w\in C} |K_a(z, w) - K(z, w)| + \sup_{z\in C}\int_{w\in\Gamma\backslash\mathbf{H}} |K_a(z, w) - K(z, w)|^2\mu_\mathbf{H}(w)\right) = 0,$$

we say that the family of functions $\{K_a\}_{a\in A}$ *converges to K in the $(\mathrm{L}^\infty_{\mathrm{loc}}, \mathrm{L}^2 \cap \mathrm{L}^\infty_{\mathrm{loc}})$-topology*. In other words, this condition means that the family converges uniformly on compact subsets of $\mathbf{H}\times\mathbf{H}$, and also with respect to the L^2-norm in the variable w, uniformly for z in compact subsets of $\Gamma\backslash\mathbf{H}$.

Theorem 2.2 (see Iwaniec [10, Theorem 7.4]). *Let $g\colon [1, \infty) \to \mathbf{R}$ be the inverse Selberg–Harish-Chandra transform of a function h as at the end of § 2.2. Then the function*

$$\begin{aligned} K_g\colon \mathbf{H} \times \mathbf{H} &\longrightarrow \mathbf{R} \\ (z, w) &\longmapsto \sum_{\gamma\in\Gamma} g(u(z, \gamma w)) \end{aligned}$$

is Γ-invariant with respect to both variables and admits the spectral representation

$$K_g(z, w) = \sum_{j=0}^{\infty} h(s_j)\phi_j(z)\bar{\phi}_j(w) + \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s=1/2} h(s)E_\mathfrak{c}(z, s)\bar{E}_\mathfrak{c}(w, s)ds, \tag{2.7}$$

where the expression on the right-hand side converges to K_g in the $(\mathrm{L}^\infty_{\mathrm{loc}}, \mathrm{L}^2 \cap \mathrm{L}^\infty_{\mathrm{loc}})$-topology.

2.4. A point counting function

We fix a real number $U \ge 1$, and define

$$\begin{aligned} g_U\colon [1,\infty) &\longrightarrow \mathbf{R} \\ u &\longmapsto \begin{cases} 1 & \text{if } u \le U; \\ 0 & \text{if } u > U. \end{cases} \end{aligned} \tag{2.8}$$

From (2.4) and the formula for $\int_1^z P_\nu(w)dw$ found in Erdélyi et al. [4, § 3.6.1, equation 8], we see that the Selberg–Harish-Chandra transform of g_U is

$$h_U(s) = 2\pi\sqrt{U^2-1}\,P_{s-1}^{-1}(U). \tag{2.9}$$

Here P_ν^μ is the Legendre function of the first kind of degree ν and order μ; see [4, § 3.2].

Now let Γ be a cofinite Fuchsian group. We introduce the following point counting function. For any two points z, w in $\mathbf{H}$ and any $U \ge 1$, we denote by $N_\Gamma(z,w,U)$ the number of translates of w by elements of Γ lying in a disc around z of radius r given by $\cosh(r) = U$, i.e.

$$\begin{aligned} N_\Gamma(z,w,U) &= \#\{\gamma \in \Gamma \mid u(z,\gamma w) \le U\} \\ &= \sum_{\gamma\in\Gamma} g_U(u(z,w)). \end{aligned} \tag{2.10}$$

This is Γ-invariant in z and w separately.

Lemma 2.3. *Let $U \in [1,3]$, and let $s \in \mathbf{C}$ be such that $s(1-s)(U-1) \in [0,1/2]$. Then $h_U(s)$ is a real number satisfying*

$$(4\pi - 8)(U-1) \le h_U(s) \le 8(U-1).$$

Proof. This follows from (2.9) and Lemma A.1. □

2.5. Bounds on eigenfunctions

The convergence of the spectral representation (2.7) can be deduced from suitable bounds on the function

$$\begin{aligned} \Phi_\Gamma\colon \mathbf{H}\times[0,\infty) &\longrightarrow [0,\infty) \\ (z,\lambda) &\longmapsto \sum_{j\colon \lambda_j\le\lambda} |\phi_j(z)|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\substack{\Re s=1/2\\ s(1-s)\le\lambda}} \bigl|E_{\mathfrak{c}}(z,s)\bigr|^2 ds. \end{aligned} \tag{2.11}$$

We will prove a bound on Φ_Γ which holds uniformly for all subgroups Γ of finite index in a given Fuchsian group Γ_0. This will give a similar uniformity in Section 4.

Lemma 2.4. *Let Γ be a cofinite Fuchsian group. Then the function $\Phi_\Gamma(z,\lambda)$ satisfies*

$$\Phi_\Gamma(z,\lambda) \le \frac{\pi}{(2\pi-4)^2} N_\Gamma(z,z,17)\lambda \quad \text{for all } z\in\mathbf{H} \text{ and all } \lambda \ge 1/4.$$

Proof. Let $z \in \mathbf{H}$ and $\lambda \ge 1/4$. We put

$$U = 1 + \frac{1}{2\lambda} \in (1,3].$$

From Bessel's inequality one can deduce (see Iwaniec [10, § 7.2]) that

$$\sum_{j\colon \lambda_j\le\lambda} |h_U(s_j)\phi_j(z)|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\substack{\Re s=1/2\\ s(1-s)\le\lambda}} \bigl|h_U(s)E_{\mathfrak{c}}(z,s)\bigr|^2 ds \le \int_{w\in\Gamma\backslash\mathbf{H}} N_\Gamma(z,w,U)^2\mu_{\mathbf{H}}(w).$$

Using the definition (2.11) of Φ_Γ and the bound $h_U(s) \ge (2\pi-4)/\lambda$ given by Lemma 2.3, we deduce

$$\Phi_\Gamma(z,\lambda) \le \frac{\lambda^2}{(2\pi-4)^2}\int_{w\in\Gamma\backslash\mathbf{H}} N_\Gamma(z,w,U)^2\mu_{\mathbf{H}}(w).$$

We rewrite the integral on the right-hand side by partial "unfolding" as follows (cf. Iwaniec [10, page 109]):

$$\begin{aligned}\int_{w\in\Gamma\backslash\mathbf{H}} N_\Gamma(z,w,U)^2\mu_{\mathbf{H}}(w) &= \sum_{\gamma,\gamma'\in\Gamma}\int_{w\in\Gamma\backslash\mathbf{H}} g_U(z,\gamma' w)g_U(\gamma z,\gamma' w)\mu_{\mathbf{H}}(w)\\ &= \sum_{\gamma\in\Gamma}\int_{w\in\mathbf{H}} g_U(z,w)g_U(\gamma z,w)\mu_{\mathbf{H}}(w).\end{aligned}$$

The last integral can be interpreted as the area of the intersection of the discs of radius r around the points z and γz of $\mathbf{H}$, where $\cosh r = U$. By the triangle inequality for the hyperbolic distance, this intersection is empty unless

$$u(z,\gamma z)\le \cosh(2r) = 2U^2-1;$$

furthermore, the area of this intersection is at most $2\pi(U-1)=\pi/\lambda$. From this we deduce that

$$\int_{w\in\Gamma\backslash\mathbf{H}} N_\Gamma(z,w,U)^2\mu_{\mathbf{H}}(w)\le\frac{\pi}{\lambda}N_\Gamma(z,z,2U^2-1)$$

Since $2U^2-1\le 17$, this proves the lemma. □

2.6. The hyperbolic lattice point problem

Let Γ be a cofinite Fuchsian group. The hyperbolic lattice point problem for Γ is the following question: what is the asymptotic behaviour of the point counting function $N_\Gamma(z,w,U)$ from (2.10) as $U\to\infty$? In contrast to the Euclidean analogue of this question, about the number of points in $\mathbf{Z}^2$ lying inside a given circle in $\mathbf{R}^2$, no elementary method is known to even give the dominant term. The difficulty is that for circles in the hyperbolic plane of radius tending to infinity, the circumference grows as fast as the enclosed area: the circumference of a circle of radius r equals $2\pi\sinh(r)$, and the area of a disc of radius r equals $2\pi(\cosh(r)-1)$. In spite of this difficulty, good estimates for $N_\Gamma(z,w,U)$ can still be found, namely using spectral theory on $\Gamma\backslash\mathbf{H}$.

The strategy is to take suitable functions

$$g_U^+, g_U^-\colon [1,\infty)\to\mathbf{R}$$

with compact support, and to define functions K_U^+ and K_U^- on $\mathbf{H}\times\mathbf{H}$, invariant with respect to the action of Γ on each of the two variables, by

$$K_U^\pm(z,w)=\sum_{\gamma\in\Gamma} g_U^\pm(u(z,\gamma w)).$$

This sum is finite because the functions $g_U^\pm$ have compact support. We take the functions $g_U^\pm$ such that for all $z,w\in\mathbf{H}$ and $U>1$, we have the inequality

$$K_U^-(z,w)\le N_\Gamma(z,w,U)\le K_U^+(z,w). \tag{2.12}$$

Provided the Selberg–Harish-Chandra transforms $h_U^\pm$ of $g_U^\pm$ satisfy the conditions of Theorem 2.2, the functions $K_U^\pm$ have spectral representations

$$K_U^\pm(z,w)=\sum_{j=0}^\infty h_U^\pm(s_j)\phi_j(z)\bar\phi_j(w)+\sum_{\mathfrak{c}}\frac{1}{4\pi i}\int_{\Re s=1/2} h_U^\pm(s)E_{\mathfrak{c}}(z,s)\bar E_{\mathfrak{c}}(w,s)ds. \tag{2.13}$$

These spectral representations can then be used to find the asymptotic behaviour of $N_\Gamma(z,w,U)$ as $U\to\infty$.

A reasonable choice at first sight would be to take for both g_U^+ and g_U^- the function g_U defined by (2.8), so that the inequalities in (2.12) become equalities. Unfortunately, the Selberg–Harish-Chandra transform h_U of g_U does not decay quickly enough as $|\Im s| \to \infty$ to give a spectral representation of $N_\Gamma(z, w, U)$ as in Theorem 2.2. We take instead

$$g_U^+(u) = \begin{cases} 1 & \text{if } 1 \le u \le U, \\ \frac{V-u}{V-U} & \text{if } U \le u \le V, \\ 0 & \text{if } V \le u \end{cases}$$

and

$$g_U^-(u) = \begin{cases} 1 & \text{if } 1 \le u \le T, \\ \frac{U-u}{U-T} & \text{if } T \le u \le U, \\ 0 & \text{if } U \le u \end{cases}$$

for certain T, V, depending on U, with $1 \le T < U < V$; see Iwaniec [10, Chapter 12]. Using (2.4), we obtain

$$\begin{aligned} h_U^+(s) &= 2\pi \int_1^V P_{s-1}(u) \frac{V-u}{V-U} du - 2\pi \int_1^V P_{s-1}(u) \frac{U-u}{V-U} du, \\ h_U^-(s) &= 2\pi \int_1^U P_{s-1}(u) \frac{U-u}{U-T} du - 2\pi \int_1^T P_{s-1}(u) \frac{T-u}{U-T} du. \end{aligned}$$

Integrating by parts and applying the integral relation between the Legendre functions P_ν and P_ν^{-2} given in Erdélyi et al. [4, § 3.6.1, equation 8], we get

$$\begin{aligned} h_U^+(s) &= 2\pi \frac{(V^2-1)P_{s-1}^{-2}(V) - (U^2-1)P_{s-1}^{-2}(U)}{V-U}, \\ h_U^-(s) &= 2\pi \frac{(U^2-1)P_{s-1}^{-2}(U) - (T^2-1)P_{s-1}^{-2}(T)}{U-T}. \end{aligned} \tag{2.14}$$

The dominant term in (2.13) as $U \to \infty$ comes from the eigenvalue $\lambda_0 = 0$, corresponding to $s_0 = 1$. It follows from (2.3) or the formula $P_0^{-2}(u) = (u-1)/(2u+2)$ that

$$h_U^+(1) = 2\pi(U-1) + \pi(V-U) \quad \text{and} \quad h_U^-(1) = 2\pi(U-1) - \pi(U-T). \tag{2.15}$$

Let a real number $\delta \ge 1$ be given. We fix parameters α^+, α^-, β^+ and β^- satisfying

$$\alpha^\pm \in (0, 1/2), \quad \beta^\pm > 0, \quad \beta^- \le \frac{\delta^{1+\alpha^-}}{\delta+1}. \tag{2.16}$$

We choose T and V as functions of U as follows:

$$T(U) = U - \beta^- U^{-1-\alpha^-}(U^2-1), \qquad V(U) = U + \beta^+ U^{-1-\alpha^+}(U^2-1). \tag{2.17}$$

The last inequality in (2.16) ensures that if $U \ge \delta$, then $T(U) \ge 1$.

For later use, we will keep the parameters $\alpha^\pm$ and $\beta^\pm$ variable for greater flexibility. To obtain the best known error bound in the hyperbolic lattice point problem, the right choice is $\alpha^\pm = 1/3$, so that

$$V - U \sim \beta^+ U^{2/3} \quad \text{and} \quad U - T \sim \beta^- U^{2/3} \quad \text{as } U \to \infty.$$

This choice leads to the following theorem.

Theorem 2.5 (Huber [9, Satz B], Patterson [15, Theorem 2], Selberg; see Iwaniec [10, Theorem 12.1]). *Let Γ be a cofinite Fuchsian group. For all $z, w \in \mathbf{H}$, the point counting function N_Γ satisfies*

$$N_\Gamma(z, w, U) = \sum_{j:\, 2/3 < s_j \le 1} 2^{s_j} \sqrt{\pi} \frac{\Gamma(s_j - \frac{1}{2})}{\Gamma(s_j + 1)} \phi_j(z) \bar{\phi}_j(w) U^{s_j} + O(U^{2/3}) \quad \text{as } U \to \infty,$$

with an implied constant depending on Γ and the points z and w.

In particular, since $|\phi_0|^2$ is the constant function $1/\mathrm{vol}_\Gamma$, this shows that

$$N_\Gamma(z,w,U) \sim \frac{2\pi(U-1)}{\mathrm{vol}_\Gamma} \quad \text{as } U \to \infty.$$

Since $2\pi(U-1)$ is the area of a disc of radius r with $\cosh r = U$, Theorem 2.5 implies that this area is asymptotically equivalent to the number of lattice points inside the disc times the area of a fundamental domain for the action of Γ, which is the intuitively expected result.

3. An approximate spectral representation of the Green function

Let Γ be a cofinite Fuchsian group. The Green function of Γ *formally* has the spectral representation

$$\mathrm{gr}_\Gamma(z,w) \stackrel{?}{=} -\sum_{j=1}^{\infty} \frac{1}{\lambda_j}\phi_j(z)\bar{\phi}_j(w) - \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s=1/2} \frac{1}{s(1-s)} E_{\mathfrak{c}}(z,s)\bar{E}_{\mathfrak{c}}(w,s)ds.$$

The problem is that this expansion does not converge. Neither should one be tempted to write the Green function by "averaging" $\mathrm{gr}_{\mathbf{H}}$ as a (likewise divergent) sum

$$\mathrm{gr}_\Gamma(z,w) \stackrel{?}{=} \sum_{\gamma\in\Gamma} \mathrm{gr}_{\mathbf{H}}(z,\gamma w).$$

However, both of these divergent expressions have at least some value as guiding ideas for what follows. In fact, we will bound $\mathrm{gr}_\Gamma(z,w)$ by means of certain functions $R^{\pm}_{\Gamma,\delta}(z,w)$, defined in (3.5) below, that reflect the above formal spectral representation of gr_Γ.

3.1. A construction of the Green function using the resolvent kernel

We will give a construction of the Green function of Γ using the family of auxiliary functions

$$\begin{aligned} g_a\colon (1,\infty) &\longrightarrow [0,\infty) \\ u &\longmapsto \frac{1}{2\pi} Q_{a-1}(u) \end{aligned}$$

for $a \ge 1$, where Q_ν is the Legendre function of the second kind of degree ν; see Erdélyi et al. [4, § 3.6.1]. By [4, § 3.6.2, equation 20], we have

$$Q_0(u) = \frac{1}{2}\log\frac{u+1}{u-1},$$

which shows that g_1 equals the function L from (1.1). By (2.4) and [4, § 3.12, equation 4], the Selberg–Harish-Chandra transform of g_a is

$$\begin{aligned} h_a(s) &= \int_1^\infty P_{s-1}(u)Q_{a-1}(u)du \\ &= \frac{1}{(a-s)(a-1+s)} \\ &= \frac{1}{s(1-s)+a(a-1)}. \end{aligned}$$

Given a real number $\sigma < 1/2$, we consider the strip

$$S_\sigma = \{s \in \mathbf{C} \mid \sigma \le \Re s \le 1-\sigma\}. \tag{3.1}$$

Lemma 3.1. *(a) For all $a, b > 1$ and all $\sigma \in (1-\min\{a,b\}, 1/2)$, the function*

$$s \mapsto \bigl|h_a(s) - h_b(s)\bigr|\bigl|s(1-s)\bigr|^2$$

is bounded on S_σ.

(b) Let $\sigma \in (0,1/2)$. There exist real numbers $(C_{a,b,\sigma})_{a,b>1}$, with $C_{a,b,\sigma} \to 0$ as both a and b tend to 1, such that

$$\bigl|h_a(s) - h_b(s)\bigr| \le C_{a,b,\sigma}\bigl|s(1-s)\bigr|^{-2} \quad \text{for all } s \in S_\sigma.$$

Proof. Both claims are easily deduced from the expression

$$h_a(s) - h_b(s) = \frac{b(b-1) - a(a-1)}{(a-s)(a-1+s)(b-s)(b+1-s)}.$$

Details are left to the reader. ∎

For all $a > 1$, the sum $\sum_{\gamma\in\Gamma} g_a(u(z,\gamma w))$ converges uniformly on compact subsets of $\mathbf{H}\times\mathbf{H}$ not containing any points of the form $(z,\gamma z)$ and defines a continuous function that is square-integrable in each variable; see Fay [7, Theorem 1.5]. We can therefore define

$$
\begin{aligned}
K_a^\Gamma\colon \{(z,w)\in\mathbf{H}\times\mathbf{H}\mid z\notin\Gamma w\} &\longrightarrow \mathbf{R}\\
(z,w) &\longmapsto \sum_{\gamma\in\Gamma} g_a(u(z,\gamma w)) - c_a,
\end{aligned}
\tag{3.2}
$$

where

$$
c_a = \frac{2\pi}{\mathrm{vol}_\Gamma}\int_1^\infty g_a(u)du = \frac{1}{\mathrm{vol}_\Gamma}h_a(1) = \frac{1}{\mathrm{vol}_\Gamma\, a(a-1)}.
$$

The constant c_a is such that the integral of K_a^Γ over $\Gamma\backslash\mathbf{H}$ with respect to each of the variables vanishes. Up to this constant, K_a^Γ is the resolvent kernel with parameter a.

It is known that the resolvent kernel admits a meromorphic continuation in the variable a. The following result can be interpreted as the statement that $-\mathrm{gr}_\Gamma$ is the constant term in the Laurent expansion of the resolvent kernel at $a = 1$.

Proposition 3.2. *The family of functions $\{-K_a^\Gamma\}_{a>1}$ converges to the Green function gr_Γ in the $(\mathrm{L}^\infty_{\mathrm{loc}}, \mathrm{L}^2\cap\mathrm{L}^\infty_{\mathrm{loc}})$-topology.*

Proof. It follows from Lemma 3.1(a) that for all $a, b > 1$, the function $g_a - g_b$ satisfies the conditions of Theorem 2.2. The function

$$
(K_a^\Gamma - K_b^\Gamma)(z,w) = \sum_{\gamma\in\Gamma}\bigl(g_a(u(z,\gamma w)) - g_b(u(z,\gamma w))\bigr) - c_a + c_b
$$

therefore has the spectral representation

$$
\begin{aligned}
(K_a^\Gamma - K_b^\Gamma)(z,w) = \sum_{j=1}^\infty (h_a(s_j) - h_b(s_j))\phi_j(z)\bar\phi_j(w)&\\
+\sum_{\mathfrak{c}}\frac{1}{4\pi i}\int_{\Re s=1/2}(h_a(s)-h_b(s))E_{\mathfrak{c}}(z,s)\bar E_{\mathfrak{c}}(w,s)ds,&
\end{aligned}
\tag{3.3}
$$

where the right-hand side converges to $K_a^\Gamma - K_b^\Gamma$ in the $(\mathrm{L}^\infty_{\mathrm{loc}}, \mathrm{L}^2\cap\mathrm{L}^\infty_{\mathrm{loc}})$-topology. (Note that the eigenvalue $\lambda_0 = 0$ has disappeared because of the definition of c_a.) In particular, $K_a^\Gamma - K_b^\Gamma$ extends to a continuous function on $\mathbf{H}\times\mathbf{H}$ that is Γ-invariant with respect to both variables.

We claim that $\{K_a^\Gamma - K_b^\Gamma\}_{a,b>1}$ converges to 0 in the $(\mathrm{L}^\infty_{\mathrm{loc}}, \mathrm{L}^2\cap\mathrm{L}^\infty_{\mathrm{loc}})$-topology as $a,b\searrow 1$. In particular, this implies that $\{K_a^\Gamma\}_{a>1}$ converges to a symmetric continuous function on $\Gamma\backslash\mathbf{H}\times\Gamma\backslash\mathbf{H}$ that is square-integrable with respect to each variable separately. We fix $\sigma\in(0,1/2)$ be such that the spectrum of $-\Delta_\Gamma$ is contained in $\{0\}\cup[\sigma(1-\sigma),\infty)$.

First we show that $\{K_a^\Gamma - K_b^\Gamma\}_{a,b>1}$ converges to zero uniformly on compact subsets of $\mathbf{H}\times\mathbf{H}$. Lemma 3.1(b) implies

$$
\begin{aligned}
\bigl|K_a^\Gamma - K_b^\Gamma\bigr|(z,w) &\le \sum_{j=1}^\infty |h_a(s_j)-h_b(s_j)|\cdot|\phi_j(z)\bar\phi_j(w)|\\
&\qquad+\sum_{\mathfrak{c}}\frac{1}{4\pi i}\int_{\Re s=1/2}|h_a(s)-h_b(s)|\cdot\bigl|E_{\mathfrak{c}}(z,s)\bar E_{\mathfrak{c}}(w,s)\bigr|ds\\
&\le C_{a,b,\sigma}\Biggl(\sum_{j=1}^\infty\bigl(s_j(1-s_j)\bigr)^{-2}|\phi_j(z)\bar\phi_j(w)|\\
&\qquad+\sum_{\mathfrak{c}}\frac{1}{4\pi i}\int_{\Re s=1/2}\bigl(s(1-s)\bigr)^{-2}\bigl|E_{\mathfrak{c}}(z,s)\bar E_{\mathfrak{c}}(w,s)\bigr|ds\Biggr).
\end{aligned}
$$

By the Cauchy–Schwarz inequality and Lemma 2.4, the right-hand side converges to 0 uniformly on compact subsets of $\mathbf{H}\times\mathbf{H}$, as claimed.

Next we show that $\{K_a^\Gamma - K_b^\Gamma\}_{a,b>1}$ converges to zero with respect to the L^2-norm in the variable w, uniformly for z in compact subsets of $\mathbf{H}$. From (3.3), Plancherel's theorem and Lemma 3.1(b), we deduce

$$\begin{aligned}\int_{w\in\Gamma\backslash\mathbf{H}} \bigl|K_a^\Gamma - K_b^\Gamma\bigr|^2(z,w)\mu_{\mathbf{H}}(w) &= \sum_{j=1}^{\infty}\bigl|h_a(s_j)-h_b(s_j)\bigr|^2|\phi_j(z)|^2 \\ &\quad + \sum_{\mathfrak{c}}\frac{1}{4\pi i}\int_{\Re s=1/2}\bigl|h_a(s)-h_b(s)\bigr|^2\bigl|E_{\mathfrak{c}}(z,s)\bigr|^2 ds \\ &\le C_{a,b,\sigma}^2\Biggl(\sum_{j=1}^{\infty}\bigl(s_j(1-s_j)\bigr)^{-4}|\phi_j(z)|^2 \\ &\qquad + \sum_{\mathfrak{c}}\frac{1}{4\pi i}\int_{\Re s=1/2}\bigl(s(1-s)\bigr)^{-4}\bigl|E_{\mathfrak{c}}(z,s)\bigr|^2 ds\Biggr)\end{aligned}$$

for real numbers $C_{a,b,\sigma}$ with $C_{a,b,\sigma}\to 0$ as $a,b\searrow 1$. By Lemma 2.4, the other factor on the right-hand side is bounded on compact subsets of $\mathbf{H}$. This implies that the right-hand side converges to 0 uniformly on compact subsets of $\mathbf{H}$ as $a,b\searrow 1$, as claimed.

The defining property (2.2) of the Selberg–Harish-Chandra transform implies that if f is a smooth, bounded, Γ-invariant function on $\mathbf{H}$, with spectral representation (2.5), then

$$\int_{w\in\Gamma\backslash\mathbf{H}} K_a^\Gamma(z,w)f(w)\mu_{\mathbf{H}}(w) = \sum_{j=1}^{\infty} b_j h_a(s_j)\phi_j(z) + \sum_{\mathfrak{c}}\frac{1}{4\pi i}\int_{\Re s=1/2} b_{\mathfrak{c}}(s)h_a(s)E_{\mathfrak{c}}(z,s)ds.$$

Taking the limit, using the L^2-convergence that we just proved and applying (2.6), we get

$$\begin{aligned}\int_{w\in\Gamma\backslash\mathbf{H}}\lim_{a\searrow 1} K_a^\Gamma(z,w)f(w)\mu_{\mathbf{H}}(w) &= \lim_{a\searrow 1}\int_{w\in\Gamma\backslash\mathbf{H}} K_a^\Gamma(z,w)f(w)\mu_{\mathbf{H}}(w) \\ &= \sum_{j=1}^{\infty}\frac{b_j}{s_j(1-s_j)}\phi_j(z) + \sum_{\mathfrak{c}}\frac{1}{4\pi i}\int_{\Re s=1/2}\frac{b_{\mathfrak{c}}(s)}{s(1-s)}E_{\mathfrak{c}}(z,s)ds \\ &= -G_\Gamma f(z) \\ &= -\int_{w\in\Gamma\backslash\mathbf{H}} \mathrm{gr}_\Gamma(z,w)f(w)\mu_{\mathbf{H}}(w).\end{aligned}$$

Since the set of smooth and bounded functions is dense in $\mathrm{L}^2(\Gamma\backslash\mathbf{H})$, this proves that the limit of the convergent family of functions $\{K_a^\Gamma\}_{a>1}$ equals $-\mathrm{gr}_\Gamma$. □

3.2. Proof of the approximate spectral representation

We now exploit the estimates for the hyperbolic lattice point problem given in § 2.6. We choose parameters $\alpha^\pm$ and $\beta^\pm$ satisfying (2.16). Using these, we define functions $T(U)$, $V(U)$, $g_U^\pm(u)$, $h_U^\pm(s)$ and $K_U^\pm(z,w)$ as in § 2.6. Furthermore, we fix a real number $\delta>1$. We write Γ as the disjoint union of subsets $\Pi_{\Gamma,\delta}(z,w)$ and $\Lambda_{\Gamma,\delta}(z,w)$ defined by

$$\begin{aligned}\Pi_{\Gamma,\delta}(z,w) &= \{\gamma\in\Gamma \mid u(z,\gamma w)\le\delta\},\\ \Lambda_{\Gamma,\delta}(z,w) &= \{\gamma\in\Gamma \mid u(z,\gamma w)>\delta\}.\end{aligned}$$

We define

$$I_\delta^\pm(s) = \frac{1}{2\pi}\int_\delta^\infty \frac{h_U^\pm(s)}{U^2-1}dU \quad \text{for } 0<\Re s<1, \tag{3.4}$$

$$R_{\Gamma,\delta}^\pm(z,w) = \sum_{j=1}^{\infty} I_\delta^\pm(s_j)\phi_j(z)\bar{\phi}_j(w) + \sum_{\mathfrak{c}}\frac{1}{4\pi i}\int_{\Re s=1/2} I_\delta^\pm(s)E_{\mathfrak{c}}(z,s)\bar{E}_{\mathfrak{c}}(w,s)ds, \tag{3.5}$$

$$q_{\Gamma,\delta}^+ = \frac{1}{\mathrm{vol}_\Gamma}\left(\frac{\beta^+}{2\alpha^+\delta^{\alpha^+}} - \log\frac{\delta+1}{2}\right),\quad q_{\Gamma,\delta}^- = -\frac{1}{\mathrm{vol}_\Gamma}\left(\frac{\beta^-}{2\alpha^-\delta^{\alpha^-}} + \log\frac{\delta+1}{2}\right). \tag{3.6}$$

The intuition behind the following theorem is that although the Green function gr_Γ does not admit a spectral representation, it can be bounded (after removing the logarithmic singularity) by functions that do admit spectral representations. The terms $q_{\Gamma,\delta}^\pm$ below correspond to the eigenvalue 0, while the terms $R_{\Gamma,\delta}^\pm(z,w)$ correspond to the non-zero part of the spectrum.

Theorem 3.3. *Let Γ be a cofinite Fuchsian group. For all $\delta > 1$ and for every choice of the parameters $\alpha^{\pm}$ and $\beta^{\pm}$ satisfying* (2.16)*, the Green function of Γ satisfies the inequalities*

$$-q^{+}_{\Gamma,\delta} - R^{+}_{\Gamma,\delta}(z,w) \leq \mathrm{gr}_{\Gamma}(z,w) + \sum_{\gamma\in\Pi_{\Gamma,\delta}(z,w)} \bigl(L(u(z,\gamma w)) - L(\delta)\bigr) \leq -q^{-}_{\Gamma,\delta} - R^{-}_{\Gamma,\delta}(z,w).$$

Proof. For any $U \geq \delta$, the inequality (2.12) implies that the number of elements $\gamma \in \Lambda_{\Gamma,\delta}(z,w)$ with $u(z,\gamma w) \leq U$ can be bounded as

$$A(U) \leq \#\{\gamma \in \Lambda_{\Gamma,\delta}(z,w) \mid u(z,\gamma w) \leq U\} \leq B(U), \tag{3.7}$$

where the functions $A, B\colon [\delta,\infty) \to \mathbf{R}$ are defined by

$$A(U) = K^{-}_{U}(z,w) - \#\Pi_{\Gamma,\delta}(z,w) \quad \text{and} \quad B(U) = K^{+}_{U}(z,w) - \#\Pi_{\Gamma,\delta}(z,w).$$

The functions A and B are continuous and increasing. The estimates from § 2.6 imply that they are bounded linearly in U as $U \to \infty$, with an implied constant depending on the group Γ, the points z and w and the functions T and V.

Let $\{h_a\}_{a>1}$, $\{g_a\}_{a>1}$ and $\{K^{\Gamma}_a\}_{a>1}$ be as in § 3.1. For all $a > 1$, applying partial summation and (3.7) gives

$$-\int_{\delta}^{\infty} g'_a(U)A(U)dU \leq \sum_{\gamma\in\Lambda_{\Gamma,\delta}(z,w)} g_a(u(z,\gamma w)) \leq -\int_{\delta}^{\infty} g'_a(U)B(U)dU.$$

Using the definition (3.2) of K^{Γ}_a, we deduce the upper bound

$$K^{\Gamma}_a(z,w) \leq \sum_{\gamma\in\Pi_{\Gamma,\delta}(z,w)} g_a(u(z,\gamma w)) - \int_{\delta}^{\infty} g'_a(U)B(U)dU - \frac{2\pi}{\mathrm{vol}_{\Gamma}}\int_{1}^{\infty} g_a(u)du.$$

The definition of B implies

$$\begin{aligned}\int_{\delta}^{\infty} g'_a(U)B(U)dU &= \int_{\delta}^{\infty} g'_a(U)K^{+}_{U}(z,w)dU - \#\Pi_{\Gamma,\delta}(z,w)\int_{\delta}^{\infty} g'_a(U)dU \\ &= \int_{\delta}^{\infty} g'_a(U)\Bigl(K^{+}_{U}(z,w) - \frac{2\pi}{\mathrm{vol}_{\Gamma}}(U-1)\Bigr)dU + \frac{2\pi}{\mathrm{vol}_{\Gamma}}\int_{\delta}^{\infty} g'_a(U)(U-1)dU \\ &\quad + \#\Pi_{\Gamma,\delta}(z,w)g_a(\delta).\end{aligned}$$

Using integration by parts, we rewrite the second integral in the last expression as follows:

$$\begin{aligned}\int_{\delta}^{\infty} g'_a(U)(U-1)dU &= \int_{1}^{\infty} g'_a(U)(U-1)dU - \int_{1}^{\delta} g'_a(U)(U-1)dU \\ &= -\int_{1}^{\infty} g_a(U)dU - \int_{1}^{\delta} g'_a(U)(U-1)dU.\end{aligned}$$

We can now rewrite our upper bound for $K^{\Gamma}_a(z,w)$ as

$$\begin{aligned}K^{\Gamma}_a(z,w) \leq \sum_{\gamma\in\Pi_{\Gamma,\delta}(z,w)} \bigl(g_a(u(z,w)) - g_a(\delta)\bigr) - \int_{\delta}^{\infty} g'_a(U)\Bigl(K^{+}_{U}(z,w) - \frac{2\pi}{\mathrm{vol}_{\Gamma}}(U-1)\Bigr)dU \\ + \frac{2\pi}{\mathrm{vol}_{\Gamma}}\int_{1}^{\delta} g'_a(U)(U-1)dU.\end{aligned}$$

Lemma A.2 implies

$$\frac{1}{2\pi}\left(\frac{2}{u+1}\right)^{a-1}\frac{1}{u^2-1} \leq g'_a(u) \leq 0,$$

and equality holds for $a = 1$. By the dominated convergence theorem, we may take the limit $a \searrow 1$ inside the integrals. Together with Proposition 3.2, this leads to

$$\begin{aligned} \mathrm{gr}_\Gamma(z,w) + \sum_{\gamma\in\Pi_{\Gamma,\delta}(z,w)} \bigl(L(u(z,\gamma w)) - L(\delta)\bigr) \ge &-\frac{1}{2\pi}\int_\delta^\infty \Bigl(K_U^+(z,w) - \frac{2\pi}{\mathrm{vol}_\Gamma}(U-1)\Bigr)\frac{dU}{U^2-1} \\ &+ \frac{1}{\mathrm{vol}_\Gamma}\log\frac{\delta+1}{2}. \end{aligned}$$

In the integral, we insert the spectral representation (2.13) of K_U^+, the formula (2.15) for $h_U^+(1)$ and the fact that $|\phi_0|^2 = 1/\mathrm{vol}_\Gamma$. We then interchange the resulting sums and integrals with the integral over U; this is permitted because the double sums and integrals converge absolutely, as one deduces from Lemma 2.4 and Theorem 2.5. This yields

$$\frac{1}{2\pi}\int_\delta^\infty \Bigl(K_U^+(z,w) - \frac{2\pi}{\mathrm{vol}_\Gamma}(U-1)\Bigr)\frac{dU}{U^2-1} = R_{\Gamma,\delta}^+(z,w) + \frac{1}{2\,\mathrm{vol}_\Gamma}\int_\delta^\infty \frac{V-U}{U^2-1}dU.$$

Finally, we note that

$$\begin{aligned} \int_\delta^\infty \frac{V-U}{U^2-1}dU &= \beta^+\int_\delta^\infty U^{-1-\alpha^+}dU \\ &= \frac{\beta^+}{\alpha^+\delta^{\alpha^+}}. \end{aligned}$$

This proves the lower bound of the theorem. The proof of the upper bound is similar. □

Remark. The only inequality responsible for the fact that the inequalities in Theorem 3.3 are not equalities is (3.7).

4. Explicit bounds

4.1. Bounds on $h_U^\pm(s)$ and $I_\delta^\pm(s)$

We keep the notation of § 3.2. In addition, we choose real numbers $\sigma^\pm$ such that

$$0 < \alpha^+ < \sigma^+ < 1/2 \quad \text{and} \quad 0 < \alpha^- < \sigma^- < 1/2.$$

Let s be in the strip S_{σ^+} defined by (3.1), and let $p_{\sigma^+}(u)$ be the elementary function defined by (A.6) below. From (2.14), Corollary A.6 and (2.17), we obtain

$$\begin{aligned} |h_U^+(s)| &\le 2\pi\frac{(V^2-1)\bigl|P_{s-1}^{-2}(V)\bigr| + (U^2-1)\bigl|P_{s-1}^{-2}(U)\bigr|}{V-U} \\ &\le 2\pi\bigl|s(1-s)\bigr|^{-5/4}\frac{p_{\sigma^+}(V)+p_{\sigma^+}(U)}{V-U} \\ &= 2\pi\bigl|s(1-s)\bigr|^{-5/4}\frac{\bigl(p_{\sigma^+}(V)+p_{\sigma^+}(U)\bigr)U^{1+\alpha^+}}{\beta^+(U^2-1)}. \end{aligned}$$

Similarly, for $s \in S_{\sigma^-}$,

$$|h_U^-(s)| \le 2\pi\bigl|s(1-s)\bigr|^{-5/4}\frac{\bigl(p_{\sigma^-}(U)+p_{\sigma^-}(T)\bigr)U^{1+\alpha^-}}{\beta^-(U^2-1)}.$$

Substituting this in the definition (3.4) of I, we obtain

$$|I_\delta^+(s)| \le D_\delta^+\bigl|s(1-s)\bigr|^{-5/4} \quad \text{and} \quad |I_\delta^-(s)| \le D_\delta^-\bigl|s(1-s)\bigr|^{-5/4}, \tag{4.1}$$

where

$$\begin{aligned} D_\delta^+ &= \frac{1}{\beta^+}\int_\delta^\infty \frac{\bigl(p_{\sigma^+}(V)+p_{\sigma^+}(U)\bigr)U^{1+\alpha^+}}{(U^2-1)^2}dU, \\ D_\delta^- &= \frac{1}{\beta^-}\int_\delta^\infty \frac{\bigl(p_{\sigma^-}(U)+p_{\sigma^-}(T)\bigr)U^{1+\alpha^-}}{(U^2-1)^2}dU. \end{aligned} \tag{4.2}$$

4.2. Bounds on gr_Γ

Theorem 4.1. *Let* Γ *be a Fuchsian group. Let* $\delta > 1$ *and* $\eta \in (0, 1/4]$ *be real numbers such that the spectrum of* $-\Delta_\Gamma$ *is contained in* $\{0\} \cup [\eta, \infty)$*. Let* σ^+, σ^-, α^+, α^-, β^+, β^- *be real numbers satisfying* (2.16) *and the inequalities*

$$0 < \alpha^+ < \sigma^+ < 1/2, \quad 0 < \alpha^- < \sigma^- < 1/2 \quad \text{and} \quad \sigma^\pm(1-\sigma^\pm) \le \eta.$$

Then the Green function of Γ *satisfies the inequalities*

$$A \le \mathrm{gr}_\Gamma(z,w) + \sum_{\gamma \in \Pi_{\Gamma,\delta}(z,w)} \bigl(L(u(z,\gamma w)) - L(\delta)\bigr) \le B \quad \text{for all } z, w \in \mathbf{H},$$

where

$$A = -q^+_{\Gamma,\delta} - D^+_\delta \frac{\pi}{(2\pi-4)^2}\left(\frac{\eta^{-5/4}}{4} + 4\sqrt{2}\right)\frac{N_\Gamma(z,z,17) + N_\Gamma(w,w,17)}{2},$$

$$B = -q^-_{\Gamma,\delta} + D^-_\delta \frac{\pi}{(2\pi-4)^2}\left(\frac{\eta^{-5/4}}{4} + 4\sqrt{2}\right)\frac{N_\Gamma(z,z,17) + N_\Gamma(w,w,17)}{2}.$$

Proof. In view of Theorem 3.3, we have to bound the absolute values of the functions $R^\pm_{\Gamma,\delta}(z,w)$ from (3.5). Applying the triangle inequality and the Cauchy–Schwarz inequality, we see that

$$\bigl|R^\pm_{\Gamma,\delta}(z,w)\bigr| \le \frac{S^\pm(z) + S^\pm(w)}{2},$$

where S^+ and S^- are defined by

$$S^\pm(z) = \sum_{j=1}^\infty |I^\pm_\delta(s_j)||\phi_j(z)|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s = 1/2} |I^\pm_\delta(s)|\bigl|E_\mathfrak{c}(z,s)\bigr|^2 ds.$$

Let $\Phi_\Gamma(z,\lambda)$ be as in (2.11). Applying (4.1), we obtain (with $\partial\Phi_\Gamma/\partial\lambda$ taken in a distributional sense)

$$\begin{aligned}
S^\pm(z)/D^\pm_\delta &\le \sum_{j=1}^\infty \lambda_j^{-5/4}|\phi_j(z)|^2 + \sum_{\mathfrak{c}} \frac{1}{4\pi i}\int_{\Re s=1/2} \bigl(s(1-s)\bigr)^{-5/4}\bigl|E_\mathfrak{c}(z,s)\bigr|^2 ds \\
&\le \eta^{-5/4} \sum_{j\colon \lambda_j \le 1/4} |\phi_j(z)|^2 + \int_{1/4}^\infty \lambda^{-5/4}\frac{\partial\Phi_\Gamma}{\partial\lambda}(z,\lambda)d\lambda \\
&= \eta^{-5/4}\Phi_\Gamma(z,1/4) + \Bigl[\lambda^{-5/4}\Phi_\Gamma(z,\lambda)\Bigr]_{\lambda=1/4}^\infty + \frac{5}{4}\int_{1/4}^\infty \lambda^{-9/4}\Phi_\Gamma(z,\lambda)d\lambda \\
&= \bigl(\eta^{-5/4} - 2^{5/2}\bigr)\Phi_\Gamma(z,1/4) + \frac{5}{4}\int_{1/4}^\infty \lambda^{-9/4}\Phi_\Gamma(z,\lambda)d\lambda.
\end{aligned}$$

The bound on $\Phi_\Gamma(z,\lambda)$ given by Lemma 2.4 implies

$$\begin{aligned}
S^\pm(z) &\le D^\pm_\delta \frac{\pi}{(2\pi-4)^2} N_\Gamma(z,z,17)\left(\bigl(\eta^{-5/4} - 2^{5/2}\bigr)\cdot\frac{1}{4} + \frac{5}{4}\int_{1/4}^\infty \lambda^{-9/4}\lambda\, d\lambda\right) \\
&= D^\pm_\delta \frac{\pi}{(2\pi-4)^2} N_\Gamma(z,z,17)\left(\frac{\eta^{-5/4}}{4} + 4\sqrt{2}\right).
\end{aligned}$$

This proves the theorem. □

Proof of Theorem 1.1. Let the notation be as in the theorem; we may assume $\eta \le 1/4$. We apply Theorem 4.1 to Γ, with parameters $\sigma^\pm$, $\alpha^\pm$ and $\beta^\pm$ depending only on η and not on Γ. It is clear that the factor $1/\mathrm{vol}_\Gamma$ occurring in the definition (3.6) is bounded by $1/\mathrm{vol}_{\Gamma_0}$, and that $N_\Gamma(z,z,17)$ is bounded by $N_{\Gamma_0}(z,z,17)$. It remains to remark that $N_{\Gamma_0}(z,z,17)$ is bounded on Y_0. □

The bounds given by Theorem 4.1 are easy to make explicit. First, the real numbers $D^\pm_\delta$ from (4.2) can be bounded in elementary ways or approximated by numerical integration. Second, a straightforward computation shows that for $z = x + iy \in \mathbf{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$, we have

$$u(z,\gamma z) = \frac{1}{2}\left((a-cx)^2 + \left(\frac{b + (a-d)x - cx^2}{y}\right)^2 + (cy)^2 + (d+cx)^2\right). \tag{4.3}$$

This can be used for concrete groups Γ to find an upper bound on $N_\Gamma(z,z,U)$ for $U > 1$.

4.3. Example: congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$

Let us consider the case where $\Gamma_0 = \mathrm{SL}_2(\mathbf{Z})$. We will make the bounds from Theorem 1.1 explicit for congruence subgroups $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$. By our convention for integration on $\Gamma\backslash\mathbf{H}$ if $-1 \in \Gamma$, we have

$$\frac{1}{\mathrm{vol}_\Gamma} \le \frac{1}{\mathrm{vol}_{\mathrm{SL}_2(\mathbf{Z})}} = \frac{\pi}{6}.$$

We choose

$$\delta = 2.$$

Selberg conjectured in [18] that the least non-zero eigenvalue λ_1 of $-\Delta_\Gamma$ is at least $1/4$, and he proved that $\lambda_1 \ge 3/16$. The sharpest result known so far, due to Kim and Sarnak [12, Appendix 2], is that $\lambda_1 \ge (25/64)(1-25/64) = 975/4096$. We may therefore take

$$\eta = 975/4096.$$

We now consider the point counting function $N_{\mathrm{SL}_2(\mathbf{Z})}(z,z,U)$ defined by (2.10) on a rectangle of the form

$$R = \{x+iy \in \mathbf{H} \mid x_{\min} \le x \le x_{\max}, y_{\min} \le y \le y_{\max}\}$$

for given real numbers $x_{\min} < x_{\max}$ and $0 < y_{\min} < y_{\max}$. The function $z \mapsto N_{\mathrm{SL}_2(\mathbf{Z})}(z,z,U)$ on R is clearly bounded from above by the number of matrices $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ such that for some $z \in R$, the inequality

$$u(z,\gamma z) \le U \tag{4.4}$$

holds. We now show how to enumerate these matrices. We distinguish the cases $c = 0$ and $c \ne 0$. We assume (after multiplying by -1 if necessary) that $a = d = 1$ in the first case, and that $c > 0$ in the second case. The total number of matrices γ as above is then twice the number produced by our enumeration.

In the case $c = 0$, by (4.3), the inequality (4.4) reduces to

$$1 + \tfrac{1}{2}(b/y)^2 \le U.$$

This implies

$$|b| \le y_{\max}\sqrt{2U-2}.$$

In the case $c > 0$, it follows from (4.3) and (4.4) that

$$\begin{gathered} |c| \le \sqrt{2U}/y_{\min}, \\ -\sqrt{2U} + cx_{\min} \le a \le \sqrt{2U} + cx_{\max}, \\ -\sqrt{2U} - cx_{\max} \le d \le \sqrt{2U} - cx_{\min}. \end{gathered}$$

Since $c \ne 0$, the coefficients a, c, d and the condition $ad - bc = 1$ determine b. If γ is a matrix obtained in this way, we compute the minimum of $u(z,\gamma z)$ for $z \in R$ using (4.3) to decide whether there exists a point $z \in R$ satisfying (4.4).

Let Y_0 denote the compact subset in $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$ which is the image of the rectangle

$$\{x+iy \in \mathbf{H} \mid -1/2 \le x \le 1/2 \text{ and } \sqrt{3}/2 \le y \le 2\}.$$

This is the complement of a disc around the unique cusp of $\mathrm{SL}_2(\mathbf{Z})$. Dividing this rectangle into 100×100 small rectangles and bounding $N_{\mathrm{SL}_2(\mathbf{Z})}(z,z,U)$ on each of them as described above, we get

$$N_{\Gamma_0}(z,z,17) \le 216 \quad \text{for all } z \in Y_0.$$

Given this upper bound for $N_{\mathrm{SL}_2(\mathbf{Z})}(z,z,17)$, some experimentation leads to the following values for the parameters:

$$\begin{gathered} \alpha^+ = 0.0366, \quad \beta^+ = 2.72, \quad \sigma^+ = 0.306, \\ \alpha^- = 2.96\cdot 10^{-3}, \quad \beta^- = 0.668, \quad \sigma^- = 0.250. \end{gathered}$$

With these choices, a numerical calculation gives

$$q^+_{\Gamma,\delta} < 69.0, \quad q^-_{\Gamma,\delta} > -216, \quad D^+_\delta < 18.5, \quad D^-_\delta < 9.61.$$

This implies the following explicit bounds on Green functions of congruence subgroups.

Theorem 4.2. *Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$. Then for all $z, w \in \mathbf{H}$ whose images in $\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H}$ lie in Y_0, we have*

$$-2.87 \cdot 10^4 \leq \mathrm{gr}_\Gamma(z, w) + \sum_{\substack{\gamma\in\Gamma \\ u(z,\gamma w)\leq 2}} \left(L(u(z, \gamma w)) - \frac{\log 3}{4\pi}\right) \leq 1.51 \cdot 10^4.$$

5. Extension to neighbourhoods of the cusps

The bounds given by Theorem 4.1 do not have the right asymptotic behaviour when z or w are near the cusps of Γ. This means that we have to do some more work to find suitable bounds on the Green function $\mathrm{gr}_\Gamma(z, w)$ in this case.

Let D and $\bar{D}$ denote the open and closed unit discs in $\mathbf{C}$, respectively. We recall that the Poisson kernel on D is defined by

$$\begin{aligned} P(\zeta) &= \frac{1 - |\zeta|^2}{|1 - \zeta|^2} \\ &= 1 + \sum_{n=1}^{\infty} \zeta^n + \sum_{n=1}^{\infty} \bar{\zeta}^n. \end{aligned}$$

We will use the notation

$$\tilde{P}(t, \zeta) = P(\exp(2\pi i t)\zeta).$$

Lemma 5.1. *The Poisson kernel satisfies*

$$\int_0^1 \tilde{P}(a, \zeta)\tilde{P}(-a, \eta)da = P(\zeta\eta) \quad \text{for all } \zeta, \eta \in D \tag{5.1}$$

and

$$\tilde{P}(t, \zeta) = \frac{d}{dt}\left(t + \frac{1}{2\pi i}\left(\log\frac{1 - \exp(-2\pi i t)\bar{\zeta}}{1 - \exp(2\pi i t)\zeta} - \log\frac{1 - \bar{\zeta}}{1 - \zeta}\right)\right) \quad \text{for all } \zeta \in D. \tag{5.2}$$

Proof. The first claim can be verified in several ways, for example using the residue theorem, Fourier series, or the fact that the Poisson kernel solves the Laplace equation with Dirichlet boundary conditions. The second claim is straightforward to check. □

Let $\mathrm{gr}_{\bar{D}}$ denote the Green function for the Laplace operator on $\bar{D}$; this is an integral kernel for the Poisson equation $\Delta f = g$ with boundary condition $f = 0$ on $\partial\bar{D}$. It is given explicitly by

$$\mathrm{gr}_{\bar{D}}(\zeta, \eta) = \frac{1}{2\pi}\log\left|\frac{\zeta - \eta}{1 - \zeta\bar{\eta}}\right| \quad \text{for all } \zeta, \eta \in D \text{ with } \zeta \neq \eta.$$

For all $\xi \in D$ and $t \in \mathbf{R}$, we write

$$\lambda(\xi, t) = \frac{1}{2\pi i}\bigl(\log(1 - \exp(-2\pi i t)\xi) - \log(1 - \exp(2\pi i t)\xi)\bigr).$$

Lemma 5.2. *The function $\lambda(\xi, t)$ satisfies*

$$\left|\int_0^t \frac{\lambda(\xi, y)}{y}dy + \frac{1}{2}\log(1 - \xi)\right| \leq \frac{1}{12t} \quad \text{for all } t > 0.$$

Proof. We expand $\lambda(\xi, t)$ for $\xi \in D$ in a Fourier series:

$$\begin{aligned} \lambda(\xi, t) &= \frac{1}{2\pi i}\left(\sum_{n=1}^{\infty}\frac{\xi^n \exp(2\pi i n t)}{n} - \sum_{n=1}^{\infty}\frac{\xi^n \exp(-2\pi i n t)}{n}\right) \\ &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\xi^n \sin(2\pi n t)}{n}. \end{aligned}$$

This implies

$$\begin{aligned}\int_0^t \frac{\lambda(\xi,y)}{y}dy &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\xi^n}{n}\int_0^t \frac{\sin(2\pi ny)}{y}dy\\ &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\xi^n}{n}\int_0^{2\pi nt}\frac{\sin x}{x}dx\\ &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\xi^n}{n}(\mathrm{si}(0)-\mathrm{si}(2\pi nt)).\end{aligned}$$

Here $\mathrm{si}(y)$ is the sine integral function normalised such that $\lim_{y\to\infty}\mathrm{si}(y)=0$:

$$\mathrm{si}(y) = \int_y^{\infty}\frac{\sin x}{x}dx.$$

It is known that

$$\mathrm{si}(0) = \frac{\pi}{2} \quad \text{and} \quad |\mathrm{si}(x)| \le \frac{1}{x} \quad \text{for all } x > 0.$$

From this we get

$$\begin{aligned}\int_0^t \frac{\lambda(\xi,y)}{y}dy &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{\xi^n}{n} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\xi^n\,\mathrm{si}(2\pi nt)}{n}\\ &= -\frac{1}{2}\log(1-\xi) - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\xi^n\,\mathrm{si}(2\pi nt)}{n}\end{aligned}$$

and

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\frac{\xi^n\,\mathrm{si}(2\pi nt)}{n}\right| &\le \sum_{n=1}^{\infty}\frac{1}{n(2\pi nt)}\\ &= \frac{1}{2\pi t}\sum_{n=1}^{\infty}\frac{1}{n^2}\\ &= \frac{1}{2\pi t}\frac{\pi^2}{6}.\end{aligned}$$

This proves the claim. ☐

For $\delta > 1$ and $u > 1$, we write

$$J_\delta(u) = \max\{0, L(u) - L(\delta)\}.$$

For $\xi \in D$, $\delta > 1$ and $\epsilon > 0$, we write

$$N_{\delta,\epsilon}(\xi) = \int_{t\in\mathbf{R}} J_\delta\left(1 + \frac{(\epsilon t)^2}{2}\right)P(\exp(2\pi i t)\xi)dt.$$

Lemma 5.3. *The function $N_{\delta,\epsilon}$ satisfies*

$$\left|N_{\delta,\epsilon}(\xi) - \frac{1}{\epsilon}\cdot\frac{2}{\pi}\arctan\sqrt{\frac{\delta-1}{2}} + \frac{1}{2\pi}\log|1-\xi|\right| \le \epsilon r_\delta \quad \textit{for all } \xi \in D,$$

where

$$r_\delta = \frac{1}{24\pi}\left(\sqrt{\frac{2}{\delta-1}} + \arctan\sqrt{\frac{\delta-1}{2}}\right). \tag{5.3}$$

Proof. We note that

$$1 + \frac{(\epsilon t)^2}{2} \le \delta \iff |t| \le \tau,$$

where

$$\tau = \frac{\sqrt{2\delta-2}}{\epsilon}.$$

By the definition of J_δ, this gives

$$N_{\delta,\epsilon}(\xi) = \int_{-\tau}^{\tau} \left(L\left(1 + \frac{(\epsilon t)^2}{2}\right) - L(\delta) \right) \tilde{P}(t,\xi) dt.$$

Using (5.2), integrating by parts, and taking the contributions for positive and negative t together, we obtain

$$\begin{aligned}
N_{\delta,\epsilon}(\xi) &= -\int_{-\tau}^{\tau} \epsilon^2 t L'(1+(\epsilon t)^2/2) \left(t + \frac{1}{2\pi i} \left(\log \frac{1-\exp(-2\pi i t)\bar{\xi}}{1-\exp(2\pi i t)\xi} - \log \frac{1-\bar{\xi}}{1-\xi} \right) \right) dt \\
&= -\int_{0}^{\tau} \epsilon^2 t L'(1+(\epsilon t)^2/2) \left(2t + \frac{1}{2\pi i} \left(\log \frac{1-\exp(-2\pi i t)\bar{\xi}}{1-\exp(2\pi i t)\xi} - \log \frac{1-\exp(2\pi i t)\bar{\xi}}{1-\exp(-2\pi i t)\xi} \right) \right) dt \\
&= -\int_{0}^{\tau} \epsilon^2 t L'(1+(\epsilon t)^2/2) \big(2t + \lambda(\xi,t) + \lambda(\bar{\xi},t) \big) dt.
\end{aligned}$$

Using the definition (1.1) of L and rearranging gives

$$\begin{aligned}
N_{\delta,\epsilon}(\xi) &= \frac{1}{2\pi} \int_0^{\tau} \epsilon^2 t \left(\frac{1}{(\epsilon t)^2} - \frac{1}{4+(\epsilon t)^2} \right) \big(2t + \lambda(\xi,t) + \lambda(\bar{\xi},t) \big) dt \\
&= \frac{1}{2\pi} \int_0^{\tau} \left(2 - \frac{2(\epsilon t)^2}{4+(\epsilon t)^2} \right) dt + \frac{1}{2\pi} \int_0^{\tau} \left(1 - \frac{(\epsilon t)^2}{4+(\epsilon t)^2} \right) \frac{\lambda(\xi,t)+\lambda(\bar{\xi},t)}{t} dt \\
&= \frac{1}{2\pi} \int_0^{\tau} \frac{8}{4+(\epsilon t)^2} dt + \frac{1}{2\pi} \int_0^{\tau} \frac{4}{4+(\epsilon t)^2} \frac{\lambda(\xi,t)+\lambda(\bar{\xi},t)}{t} dt.
\end{aligned} \tag{5.4}$$

We consider the two integrals in the last expression one by one. As for the first integral, we have

$$\begin{aligned}
\int_0^{\tau} \frac{8}{4+(\epsilon t)^2} dt &= \frac{2}{\epsilon} \int_0^{\epsilon\tau/2} \frac{2}{1+x^2} dx \\
&= \frac{4}{\epsilon} \arctan \frac{\epsilon \tau}{2} \\
&= \frac{4}{\epsilon} \arctan \sqrt{\frac{\delta - 1}{2}}.
\end{aligned} \tag{5.5}$$

As for the second integral in (5.4), let us write for convenience

$$I_\xi = \int_0^{\tau} \frac{4}{4+(\epsilon t)^2} \frac{\lambda(\xi,t)+\lambda(\bar{\xi},t)}{t} dt$$

and

$$\Lambda_\xi(t) = \int_0^{t} \frac{\lambda(\xi,y)+\lambda(\bar{\xi},y)}{y} dy + \log|1-\xi|.$$

Then we have

$$\Lambda_\xi'(t) = \frac{\lambda(\xi,t)+\lambda(\bar{\xi},t)}{t} \quad \text{and} \quad \Lambda_\xi(0) = \log|1+\xi|.$$

Integration by parts gives

$$I_\xi = -\log|1-\xi| + \frac{4}{4+(\epsilon\tau)^2} \Lambda_\xi(\tau) + \int_0^{\tau} \frac{8\epsilon^2 t}{(4+(\epsilon t)^2)^2} \Lambda_\xi(t) dt.$$

By Lemma 5.2, it follows that

$$\big| I_\xi + \log|1-\xi| \big| \leq + \frac{4}{4+(\epsilon\tau)^2} \frac{1}{6\tau} + \int_0^{\tau} \frac{8\epsilon^2 t}{(4+(\epsilon t)^2)^2} \frac{1}{6t} dt.$$

The integral can be evaluated by elementary means, and the result is

$$\begin{aligned}
\big| I_\xi + \log|1-\xi| \big| &\leq \frac{1}{6\tau} + \frac{\epsilon}{12} \arctan \frac{\epsilon\tau}{2} \\
&= \frac{\epsilon}{12} \left(\sqrt{\frac{2}{\delta-1}} + \arctan \sqrt{\frac{\delta-1}{2}} \right).
\end{aligned}$$

Combining this with (5.4) and (5.5) proves the claim. □

Lemma 5.4. *Let Γ be a cofinite Fuchsian group, and let $\delta > 1$ and $\epsilon' > \epsilon > 0$ be real numbers satisfying the inequalities*

$$\bigl(\delta + \sqrt{\delta^2 - 1}\bigr)^{1/2}\epsilon' \le \min_{\substack{\gamma\in\Gamma\\ \gamma\notin\Gamma_{\mathfrak{c}}}} C_{\mathfrak{c}}(\gamma) \quad \text{and} \quad \bigl(\delta + \sqrt{\delta^2 - 1}\bigr)\epsilon \le \epsilon'.$$

(a) For all $z, w \in \mathbf{H}$ with $y_{\mathfrak{c}}(z) \ge 1/\epsilon'$ and $y_{\mathfrak{c}}(w) \ge 1/\epsilon'$ and all $\gamma \in \Gamma$, we have

$$u(z, \gamma w) < \delta \implies \gamma \in \Gamma_{\mathfrak{c}}.$$

(b) For all $z, w \in \mathbf{H}$ such that $y_{\mathfrak{c}}(z) \ge 1/\epsilon$ and such that the image of w in $\Gamma\backslash\mathbf{H}$ lies outside $D_{\mathfrak{c}}(\epsilon')$, and for all $\gamma \in \Gamma$, we have $u(z, \gamma w) \ge \delta$.

Proof. Let z, w and γ be as in (a). We write

$$\sigma_{\mathfrak{c}}^{-1}\gamma\sigma_{\mathfrak{c}} = \gamma' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose $\gamma \notin \Gamma_{\mathfrak{c}}$. Then our assumptions imply

$$|c|/\epsilon' \ge \bigl(\delta + \sqrt{\delta^2 - 1}\bigr)^{1/2}. \tag{5.6}$$

We have

$$\begin{aligned} u(z, \gamma w) &= u(\sigma_{\mathfrak{c}}^{-1} z, \sigma_{\mathfrak{c}}^{-1}\gamma w) \\ &= u(\sigma_{\mathfrak{c}}^{-1} z, \gamma'\sigma_{\mathfrak{c}}^{-1} w) \\ &= 1 + \frac{|\sigma_{\mathfrak{c}}^{-1} z - \gamma'\sigma_{\mathfrak{c}}^{-1} w|^2}{2(\Im\sigma_{\mathfrak{c}}^{-1} z)(\Im\gamma'\sigma_{\mathfrak{c}}^{-1} w)} \\ &\ge 1 + \frac{(\Im\sigma_{\mathfrak{c}}^{-1} z - \Im\gamma'\sigma_{\mathfrak{c}}^{-1} w)^2}{2(\Im\sigma_{\mathfrak{c}}^{-1} z)(\Im\gamma'\sigma_{\mathfrak{c}}^{-1} w)} \\ &= \frac{1}{2}\biggl(\frac{\Im\sigma_{\mathfrak{c}}^{-1} z}{\Im\gamma'\sigma_{\mathfrak{c}}^{-1} w} + \frac{\Im\gamma'\sigma_{\mathfrak{c}}^{-1} w}{\Im\sigma_{\mathfrak{c}}^{-1} z}\biggr) \\ &= \frac{1}{2}\biggl(\frac{y_{\mathfrak{c}}(z)|c\sigma_{\mathfrak{c}}^{-1} w + d|^2}{y_{\mathfrak{c}}(w)} + \frac{y_{\mathfrak{c}}(w)}{y_{\mathfrak{c}}(z)|c\sigma_{\mathfrak{c}}^{-1} w + d|^2}\biggr). \end{aligned}$$

From (5.6), we deduce

$$\begin{aligned} \frac{y_{\mathfrak{c}}(z)|c\sigma_{\mathfrak{c}}^{-1} w + d|^2}{y_{\mathfrak{c}}(w)} &\ge \frac{y_{\mathfrak{c}}(z)(c y_{\mathfrak{c}}(w))^2}{y_{\mathfrak{c}}(w)} \\ &= c^2 y_{\mathfrak{c}}(z) y_{\mathfrak{c}}(w) \\ &\ge (|c|/\epsilon')^2 \\ &\ge \delta + \sqrt{\delta^2 - 1}. \end{aligned}$$

Using the fact that the function $x \mapsto x + x^{-1}$ is increasing for $x \ge 1$, we obtain

$$\begin{aligned} u(z, \gamma w) &\ge \frac{1}{2}\biggl(\bigl(\delta + \sqrt{\delta^2 - 1}\bigr) + \frac{1}{\delta + \sqrt{\delta^2 - 1}}\biggr) \\ &= \delta. \end{aligned}$$

This proves (a).

Now let z, w and γ be as in (b). Our assumption that the image of w in $\Gamma\backslash\mathbf{H}$ lies outside $D_{\mathfrak{c}}(\epsilon')$ implies

$$y_{\mathfrak{c}}(\gamma w) \le 1/\epsilon'$$

and hence

$$\begin{aligned} \frac{y_{\mathfrak{c}}(z)}{y_{\mathfrak{c}}(\gamma w)} &\ge \frac{\epsilon'}{\epsilon} \\ &\ge \delta + \sqrt{\delta^2 - 1}. \end{aligned}$$

Using the fact that the function $x \mapsto x^{-1}$ is increasing for $x \geq 1$ as in the proof of (a), we get

$$\begin{aligned} u(z, \gamma w) &= u(\sigma_{\mathfrak{c}}^{-1} z, \sigma_{\mathfrak{c}}^{-1} \gamma w) \\ &\geq \frac{1}{2}\left(\frac{y_{\mathfrak{c}}(z)}{y_{\mathfrak{c}}(\gamma w)} + \frac{y_{\mathfrak{c}}(\gamma w)}{y_{\mathfrak{c}}(z)}\right) \\ &\geq \frac{1}{2}\left(\bigl(\delta + \sqrt{\delta^2 - 1}\bigr) + \frac{1}{\delta + \sqrt{\delta^2 - 1}}\right) \\ &= \delta. \end{aligned}$$

This proves (b). $\square$

In the following proposition, we extend our bounds on gr_Γ to the neighbourhoods $D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$ of the cusps. We make the following abuse of notation: for $z \in \mathbf{H}$ and S a subset of $\Gamma\backslash\mathbf{H}$, we write $z \in S$ if the image of z in $\Gamma\backslash\mathbf{H}$ lies in S.

Proposition 5.5. *Let Γ be a cofinite Fuchsian group, and let δ be a real number with $\delta > 1$. For every cusp $\mathfrak{c}$ of Γ, let $\epsilon'_{\mathfrak{c}} > \epsilon_{\mathfrak{c}} > 0$ be real numbers satisfying the inequalities*

$$\epsilon'_{\mathfrak{c}}\bigl(\delta + \sqrt{\delta^2 - 1}\bigr)^{1/2} \leq \min_{\substack{\gamma\in\Gamma \\ \gamma\notin\Gamma_{\mathfrak{c}}}} C_{\mathfrak{c}}(\gamma), \quad \text{and} \quad \bigl(\delta + \sqrt{\delta^2 - 1}\bigr)\epsilon_{\mathfrak{c}} \leq \epsilon'_{\mathfrak{c}}$$

and small enough such that the discs $D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$ are pairwise disjoint. Let

$$Y = (\Gamma\backslash\mathbf{H}) \setminus \bigsqcup_{\mathfrak{c}} D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}}).$$

Let A and B be real numbers satisfying

$$A \leq \mathrm{gr}_\Gamma(z, w) + \sum_{\substack{\gamma\in\Gamma \\ u(z,\gamma w)\leq\delta}} \bigl(L(u(z, w)) - L(\delta)\bigr) \leq B \quad \text{for all } z, w \in Y. \tag{5.7}$$

(a) If $\mathfrak{c}$ is a cusp such that $z \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$, $w \in Y$ and $w \notin D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$, we have

$$A \leq \mathrm{gr}_\Gamma(z, w) - \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon_{\mathfrak{c}} y_{\mathfrak{c}}(z)) \leq B.$$

(a′) If $\mathfrak{c}$ is a cusp such that $w \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$, $z \in Y$ and $z \notin D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$, we have

$$A \leq \mathrm{gr}_\Gamma(z, w) - \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon_{\mathfrak{c}} y_{\mathfrak{c}}(w)) \leq B.$$

(b) If $\mathfrak{c}$, $\mathfrak{d}$ are two distinct cusps such that $z \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$ and $w \in D_{\mathfrak{d}}(\epsilon_{\mathfrak{d}})$, we have

$$A \leq \mathrm{gr}_\Gamma(z, w) - \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon_{\mathfrak{c}} y_{\mathfrak{c}}(z)) - \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon_{\mathfrak{c}} y_{\mathfrak{c}}(w)) \leq B.$$

(c) If $\mathfrak{c}$ is a cusp such that $z, w \in D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$, we have

$$\begin{aligned} \tilde{A}_{\mathfrak{c}} \leq \mathrm{gr}_\Gamma(z, w) &- \#(\Gamma \cap \{\pm 1\}) \cdot \frac{1}{2\pi} \log|q_{\mathfrak{c}}(z) - q_{\mathfrak{c}}(w)| \\ &- \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon'_{\mathfrak{c}} y_{\mathfrak{c}}(z)) - \frac{1}{\mathrm{vol}_\Gamma} \log(\epsilon'_{\mathfrak{c}} y_{\mathfrak{c}}(w)) \leq \tilde{B}_{\mathfrak{c}}, \end{aligned}$$

where $\tilde{A}_{\mathfrak{c}}$ and $\tilde{B}_{\mathfrak{c}}$ are defined using the function r_δ from (5.3) by

$$\begin{aligned} \tilde{A}_{\mathfrak{c}} &= A + \#(\Gamma \cap \{\pm 1\})\left[\frac{1}{\epsilon'_{\mathfrak{c}}}\left(1 - \frac{2}{\pi}\arctan\sqrt{\frac{\delta - 1}{2}}\right) - \epsilon'_{\mathfrak{c}} r_\delta\right], \\ \tilde{B}_{\mathfrak{c}} &= B + \#(\Gamma \cap \{\pm 1\})\left[\frac{1}{\epsilon'_{\mathfrak{c}}}\left(1 - \frac{2}{\pi}\arctan\sqrt{\frac{\delta - 1}{2}}\right) + \epsilon'_{\mathfrak{c}} r_\delta\right]. \end{aligned}$$

Proof. In view of § 2.1 (or Lemma 5.4(a)), the discs $D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}}) \supset D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$ are well defined. Furthermore, the assumption that the discs $D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$ are pairwise disjoint implies that for every cusp $\mathfrak{c}$, the boundaries of $\bar{D}_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$ and $\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$ are contained in Y.

Let us prove part (a). We keep $w \in Y$ fixed and consider $\mathrm{gr}_\Gamma(z,w)$ as a function of $z \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$. The defining properties of gr_Γ imply

$$\mathrm{gr}_\Gamma(z,w) = \frac{1}{\mathrm{vol}_\Gamma}\log(\epsilon_{\mathfrak{c}} y_{\mathfrak{c}}(z)) + h_w(z) \quad \text{for all } z \in D_{\mathfrak{c}}(\epsilon_{\mathfrak{c}}),$$

with h_w a real-valued harmonic function on $\bar{D}_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$. By construction, $h_w(z)$ coincides with $\mathrm{gr}_\Gamma(z,w)$ for z on the boundary of $\bar{D}_{\mathfrak{c}}(\epsilon_{\mathfrak{c}})$. This implies

$$h_w(z) = \int_0^1 \mathrm{gr}_\Gamma(\sigma_{\mathfrak{c}}(a + i/\epsilon_{\mathfrak{c}}), w)\tilde{P}(-a, q_{\mathfrak{c}}(z)\exp(2\pi/\epsilon_{\mathfrak{c}}))da.$$

By Lemma 5.4(b), there are no $\gamma \in \Gamma$ such that $u(z,\gamma w) < \delta$. By the assumption (5.7) on A and B, we conclude

$$A \le h_w(z) \le B \quad \text{for all } z \in D_{\mathfrak{c}}(\epsilon).$$

This proves (a). Part (a′) is equivalent to (a) by symmetry, and (b) is proved in a similar way.

It remains to prove part (c). We identify $\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$ with the closed unit disc $\bar{D}$ via the map

$$\begin{aligned} \bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}}) &\xrightarrow{\sim} \bar{D} \\ z &\longmapsto \zeta_z = q_{\mathfrak{c}}(z)\exp(2\pi/\epsilon'_{\mathfrak{c}}). \end{aligned}$$

Let $\mathrm{gr}_{\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})}$ be the Green function for the Laplace operator on $\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$, given in terms of gr_D by

$$\mathrm{gr}_{\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})}(z,w) = \#(\Gamma \cap \{\pm 1\}) \cdot \mathrm{gr}_D(\zeta_z, \zeta_w).$$

The factor $\#(\Gamma \cap \{\pm 1\})$ arises because of how we defined integration on $\Gamma\backslash\mathbf{H}$ in Section 1.

Fixing w and considering gr_Γ as a function of z, we have

$$\mathrm{gr}_\Gamma(z,w) = \mathrm{gr}_{\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})}(z,w) + \frac{1}{\mathrm{vol}_\Gamma}\log(\epsilon'_{\mathfrak{c}} y_{\mathfrak{c}}(z)) + h_w(z) \quad \text{for all } z \in D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}}),$$

where h_w is a real-valued harmonic function on $\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$. By construction, $h_w(z)$ coincides with $\mathrm{gr}_\Gamma(z,w)$ for z on the boundary of $\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$. This implies

$$h_w(z) = \int_0^1 \mathrm{gr}_\Gamma(\sigma_{\mathfrak{c}}(a + i/\epsilon'_{\mathfrak{c}}), w)\tilde{P}(-a, \zeta_z)da.$$

Applying the same argument to $\mathrm{gr}_\Gamma(\sigma_{\mathfrak{c}}(a + i/\epsilon'_{\mathfrak{c}}), w)$ as a function of w, we obtain

$$\mathrm{gr}_\Gamma(z,w) = \mathrm{gr}_{\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})}(z,w) + \frac{1}{\mathrm{vol}_\Gamma}\log(\epsilon'_{\mathfrak{c}} y_{\mathfrak{c}}(z)) + \frac{1}{\mathrm{vol}_\Gamma}\log(\epsilon'_{\mathfrak{c}} y_{\mathfrak{c}}(w)) + K(z,w), \tag{5.8}$$

where K is the function on $D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}}) \times D_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})$ defined by

$$K(z,w) = \int_{a=0}^1 \int_{b=0}^1 \mathrm{gr}_\Gamma(\sigma_{\mathfrak{c}}(a + i/\epsilon'_{\mathfrak{c}}), \sigma_{\mathfrak{c}}(b + i/\epsilon'_{\mathfrak{c}}))\tilde{P}(-b, \zeta_w)\tilde{P}(-a, \zeta_z)db\,da.$$

In (5.7), we may replace Γ by $\Gamma_{\mathfrak{c}}$ in view of Lemma 5.4(a), i.e., we have

$$A \le \mathrm{gr}_\Gamma(z,w) + \sum_{\substack{\gamma \in \Gamma_{\mathfrak{c}} \\ u(z,\gamma w) \le \delta}} \bigl(L(u(z,\gamma w)) - L(\delta)\bigr) \le B \quad \text{for all } z, w \in \partial\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}}).$$

Substituting this in the definition of $K(z,w)$ and "unfolding" the action of $\Gamma_{\mathfrak{c}}$, we get

$$A \le K(z,w) + \#(\Gamma \cap \{\pm 1\}) M(\zeta_z, \zeta_w) \le B, \tag{5.9}$$

where M is the function on $D \times D$ defined by

$$M(\zeta,\eta) = \int_{a=0}^{1} \int_{b\in\mathbf{R}} J_\delta(u(\sigma_{\mathfrak{c}}(a+i/\epsilon'_{\mathfrak{c}}), \sigma_{\mathfrak{c}}(b+i/\epsilon'_{\mathfrak{c}}))) \tilde{P}(-b,\eta)\tilde{P}(-a,\zeta) db\, da.$$

Making the change of variables

$$b = a + t$$

and noting that

$$\begin{aligned} u(\sigma_{\mathfrak{c}}(a+i/\epsilon'_{\mathfrak{c}}), \sigma_{\mathfrak{c}}(b+i/\epsilon'_{\mathfrak{c}})) &= u(a+i/\epsilon'_{\mathfrak{c}}, b+i/\epsilon'_{\mathfrak{c}}) \\ &= 1 + \frac{(b-a)^2}{2/\epsilon'^2_{\mathfrak{c}}} \\ &= 1 + \frac{(\epsilon'_{\mathfrak{c}} t)^2}{2}, \end{aligned}$$

we obtain

$$M(\zeta,\eta) = \int_{a=0}^{1} \int_{t\in\mathbf{R}} J_\delta\biggl(1 + \frac{(\epsilon'_{\mathfrak{c}} t)^2}{2}\biggr) \tilde{P}(-a-t,\eta)\tilde{P}(-a,\zeta) dt\, da.$$

Interchanging the order of integration, noting that

$$\tilde{P}(-a-t,\eta) = \tilde{P}(a, \exp(2\pi i t)\bar{\eta})$$

and using (5.1), we simplify this to

$$\begin{aligned} M(\zeta,\eta) &= \int_{t\in\mathbf{R}} J_\delta\biggl(1 + \frac{(\epsilon'_{\mathfrak{c}} t)^2}{2}\biggr) \tilde{P}(t,\zeta\bar{\eta}) dt \\ &= N_{\delta,\epsilon'_{\mathfrak{c}}}(\zeta\bar{\eta}). \end{aligned}$$

Applying Lemma 5.3, we conclude from (5.9) that

$$\begin{aligned} K(z,w) &\le B + \#(\Gamma\cap\{\pm 1\})\biggl(\frac{1}{2\pi}\log|1-\zeta_z\bar{\zeta}_w| - \frac{1}{\epsilon'_{\mathfrak{c}}}\cdot\frac{2}{\pi}\arctan\sqrt{\frac{\delta-1}{2}} + \epsilon'_{\mathfrak{c}} r_\delta\biggr), \\ K(z,w) &\ge A + \#(\Gamma\cap\{\pm 1\})\biggl(\frac{1}{2\pi}\log|1-\zeta_z\bar{\zeta}_w| - \frac{1}{\epsilon'_{\mathfrak{c}}}\cdot\frac{2}{\pi}\arctan\sqrt{\frac{\delta-1}{2}} - \epsilon'_{\mathfrak{c}} r_\delta\biggr). \end{aligned}$$

We now note that

$$\begin{aligned} \mathrm{gr}_{\bar{D}_{\mathfrak{c}}(\epsilon'_{\mathfrak{c}})}(z,w) &= \#(\Gamma\cap\{\pm 1\}) \cdot \frac{1}{2\pi}\log\left|\frac{\zeta_z - \zeta_w}{1-\zeta_z\bar{\zeta}_w}\right| \\ &= \#(\Gamma\cap\{\pm 1\}) \left[\frac{1}{2\pi}\log|q_{\mathfrak{c}}(z) - q_{\mathfrak{c}}(w)| + \frac{1}{\epsilon'_{\mathfrak{c}}} - \frac{1}{2\pi}\log|1-\zeta_z\bar{\zeta}_w|\right]. \end{aligned}$$

Combining this with (5.8) and the above bounds on $K(z,w)$ yields the proposition. □

Appendix: Bounds on Legendre functions

In this appendix, we prove a number of bounds on the Legendre functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ that are used in the rest of the paper.

Lemma A.1. *Let $u \in [1,3]$ and $\lambda \geq 0$ be such that $\lambda(u-1) \leq \frac{1}{2}$, and let $s \in \mathbf{C}$ be such that $s(1-s) = \lambda$. Then the real number $P_{s-1}^{-1}(u)$ satisfies the inequalities*

$$(2 - 4/\pi)\sqrt{\frac{u-1}{u+1}} \leq P_{s-1}^{-1}(u) \leq (4/\pi)\sqrt{\frac{u-1}{u+1}}.$$

Proof. We start by expressing the Legendre function P_ν^μ in terms of Gauß's hypergeometric function $F(a,b;c;z)$. Because of the many symmetries satisfied by the hypergeometric function (see Erdélyi et al. [4, Chapter II]), there are lots of ways to do this. Using [4, § 3.2, equation 3] gives

$$P_{s-1}^{-1}(u) = \sqrt{\frac{u-1}{u+1}} F\left(s, 1-s; 2; \frac{1-u}{2}\right).$$

Next we use the hypergeometric series for $F(a,b;c;z)$ with $z < 1$ (see [4, § 2.1, equation 2]):

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{A.1}$$

where

$$(y)_n = \Gamma(y+n)/\Gamma(y) = y(y+1)\cdots(y+n-1).$$

Putting $x = \frac{u-1}{2}$ for a moment, using (A.1) and applying the triangle inequality, we get the bound

$$\bigl|F(s,1-s;2;-x) - 1\bigr| \leq \sum_{n\geq 1} \left| \frac{(s)_n (1-s)_n}{(2)_n n!} (-x)^n \right|$$

The assumption $\lambda(u-1) \leq \frac{1}{2}$ is equivalent to $\lambda x \leq \frac{1}{4}$. Therefore the n-th term in the series on the right-hand side can be bounded as follows:

$$\begin{aligned}
\left| \frac{(s)_n (1-s)_n}{(2)_n n!} (-x)^n \right| &= \frac{\prod_{k=0}^{n-1} \bigl(s(1-s)x + k(k+1)x\bigr)}{(2)_n n!} \\
&\leq \frac{\prod_{k=0}^{n-1} \bigl(\frac{1}{4} + k(k+1)\bigr)}{(2)_n n!} \\
&= \frac{\bigl(\frac{1}{2}\bigr)_n \bigl(\frac{1}{2}\bigr)_n}{(2)_n n!}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\bigl|F(s,1-s;2;-x) - 1\bigr| &\leq F\bigl(\tfrac{1}{2}, \tfrac{1}{2}; 2; 1\bigr) - 1 \\
&= 4/\pi - 1,
\end{aligned}$$

where the last equality follows from the formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{for } \Re c > 0 \text{ and } \Re c > \Re(a+b) \tag{A.2}$$

(see Erdélyi et al. [4, § 2.1.3, equation 14] or Iwaniec [10, equation B.20]) and the fact that $\Gamma(3/2) = \sqrt{\pi}/2$. We conclude that

$$\begin{aligned}
\left| P_{s-1}^{-1} - \sqrt{\frac{u-1}{u+1}} \right| &= \sqrt{\frac{u-1}{u+1}} \bigl|F(s,1-s;2;-x) - 1\bigr| \\
&\leq \sqrt{\frac{u-1}{u+1}} (4/\pi - 1) \\
&= (4/\pi - 1)\sqrt{\frac{u-1}{u+1}},
\end{aligned}$$

which is equivalent to the inequalities in the statement of the lemma. □

Lemma A.2. *For all real numbers $\nu \geq 0$ and $u > 1$, the real number $Q'_\nu(u)$ satisfies*

$$\left(\frac{2}{u+1}\right)^\nu \frac{1}{u^2-1} \leq Q'_\nu(u) \leq 0.$$

Proof. We express Q'_ν in terms of the hypergeometric function using [4, § 3.6.1, equation 5, and § 3.2, equation 36]:

$$Q'_\nu(u) = -\left(\frac{2}{u+1}\right)^\nu \frac{1}{u^2-1} \frac{\Gamma(1+\nu)\Gamma(2+\nu)}{\Gamma(2+2\nu)} F\left(\nu, 1+\nu; 2+2\nu; \frac{2}{u+1}\right).$$

Since $2/(u+1) < 1$, the hypergeometric function is given by the series (A.1). The non-negativity of all the arguments gives the bounds

$$\begin{aligned} 0 &\leq F\left(\nu, 1+\nu; 2+2\nu; \frac{2}{u+1}\right) \\ &\leq F(\nu, 1+\nu; 2+2\nu; 1) \\ &= \frac{\Gamma(2+2\nu)\Gamma(1)}{\Gamma(2+\nu)\Gamma(1+\nu)}; \end{aligned}$$

the last equality follows from (A.2). Combining this with the above formula for $Q'_\nu(u)$ yields the claim. □

Lemma A.3. *For all real numbers a, b and y with $b \geq a > 0$, we have*

$$\begin{aligned} \exp\left(-\frac{1}{12}\left(\frac{1}{a} - \frac{1}{b}\right)\right) \frac{(a^2+y^2)^{a/2-1/4}}{(b^2+y^2)^{b/2-1/4}} &\leq \left|\frac{\Gamma(a+iy)}{\Gamma(b+iy)}\right| \\ &\leq \exp\left(b - a + \frac{1}{12}\left(\frac{1}{a} - \frac{1}{b}\right)\right) \frac{(a^2+y^2)^{a/2-1/4}}{(b^2+y^2)^{b/2-1/4}}. \end{aligned}$$

Proof. We use Binet's formula for $\log\Gamma$ (see Erdélyi et al. [4, § 1.9, equation 4]):

$$\log\Gamma(z) = \left(z - \tfrac{1}{2}\right)\log z - z + \tfrac{1}{2}\log 2\pi + \int_0^\infty B(t)\exp(-zt)dt \quad \text{for } \Re z > 0,$$

where

$$\begin{aligned} B(t) &= \frac{(\exp(t)-1)^{-1} - t^{-1} + \frac{1}{2}}{t} \\ &= \frac{\frac{t}{2}\coth\frac{t}{2} - 1}{t^2}. \end{aligned}$$

We write

$$\begin{aligned} M(a,y) &= \Re\int_0^\infty B(t)\exp(-(a+iy)t)dt \\ &= \int_0^\infty B(t)\exp(-at)\cos(yt)dt. \end{aligned}$$

Then we have

$$\begin{aligned} \log\left|\frac{\Gamma(a+iy)}{\Gamma(b+iy)}\right| &= \Re\Big((a+iy-1/2)\log(a+iy) - (b+iy-1/2)\log(b+iy)\Big) \\ &\quad - a + b + M(a,y) - M(b,y) \\ &= (a-1/2)\log|a+iy| - (b-1/2)\log|b+iy| \\ &\quad - y\arg(a+iy) + y\arg(b+iy) - a + b + M(a,y) - M(b,y). \end{aligned}$$

We note that

$$y\arg(a+iy) - y\arg(b+iy) = y\arctan\frac{(b-a)y}{ab+y^2} \in [0, b-a].$$

Using $b \geq a$, we conclude

$$\log \left| \frac{\Gamma(a+iy)}{\Gamma(b+iy)} \right| \leq (a-1/2)\log|a+iy| - (b-1/2)\log|b+iy| - a + b + M(a,y) - M(b,y)$$

and

$$\log \left| \frac{\Gamma(a+iy)}{\Gamma(b+iy)} \right| \geq (a-1/2)\log|a+iy| - (b-1/2)\log|b+iy| + M(a,y) - M(b,y).$$

It remains to bound $M(a,y) - M(b,y)$. The function B satisfies

$$0 < B(t) \leq \lim_{x\to 0} B(x) = 1/12 \quad \text{for all } t > 0.$$

Using this and the positivity of $\exp(-at) - \exp(-bt)$, we bound $M(a,y) - M(b,y)$ as follows:

$$\begin{aligned}
\bigl|M(a,y) - M(b,y)\bigr| &\leq \int_0^\infty B(t)\bigl(\exp(-at) - \exp(-bt)\bigr)\,|\cos(yt)|\,dt \\
&= \int_0^\infty B(t)\bigl(\exp(-at) - \exp(-bt)\bigr)dt \\
&\leq \frac{1}{12}\int_0^\infty \bigl(\exp(-at) - \exp(-bt)\bigr)dt \\
&= \frac{1}{12}\left(\frac{1}{a} - \frac{1}{b}\right).
\end{aligned}$$

This implies the inequality we wanted to prove. □

Corollary A.4. *For $b \geq a > 0$, $b \geq 1/2$, and $y \in \mathbf{R}$, we have*

$$\left| \frac{\Gamma(a+iy)}{\Gamma(b+iy)} \right| \leq \exp\left(b - a + \frac{1}{12}\left(\frac{1}{a} - \frac{1}{b}\right)\right)(a^2+y^2)^{-(b-a)/2}.$$

Given a real number $\sigma \in (0, 1/2)$, we consider the strip

$$S_\sigma = \{s \in \mathbf{C} \mid \sigma \leq \Re s \leq 1 - \sigma\}.$$

We put

$$\begin{aligned}
C_\sigma &= \max\{1, \tan\pi\sigma\}(\sigma^{-1} - 1)^{1/4}\exp\left(\frac{1}{2} + \frac{1}{24\sigma(\frac{1}{2}+\sigma)}\right), \\
C'_\sigma &= \max\{1, \tan\pi\sigma\}(\sigma^{-1} - 1)^{1/4}\exp\left(\frac{1}{2} + \frac{1}{24(1-\sigma)(\frac{3}{2}-\sigma)}\right).
\end{aligned} \tag{A.3}$$

Proposition A.5. *Let m be an even non-negative integer, and let $\sigma \in (0, 1/2)$. For all $s \in S_\sigma$ and all $u > 1$, we have*

$$\bigl|P_{s-1}^{-m}(u)\bigr| \leq \bigl|s(1-s)\bigr|^{-(2m+1)/4}\frac{C_\sigma x^{m-\sigma} + C'_\sigma x^{m-1+\sigma}}{\sqrt{\pi}(4(u^2-1))^{m/2}}\sum_{n=0}^\infty \frac{|(\frac{1}{2}-m)_n|}{n!}x^{-2n},$$

where

$$x = u + \sqrt{u^2-1}, \quad u = \frac{x + x^{-1}}{2}.$$

Proof. We use the following expression for P_{s-1}^{-m} (see Erdélyi et al. [4, § 3.2, equation 27]):

$$\begin{aligned}
P_{s-1}^{-m}(u) = {}& \frac{\Gamma(-\frac{1}{2}+s)}{\sqrt{\pi}\Gamma(m+s)}\frac{x^{m+s-1}}{(x-x^{-1})^m}F\bigl(\tfrac{1}{2}-m, 1-m-s; \tfrac{3}{2}-s; x^{-2}\bigr) \\
&+ \frac{\Gamma(\frac{1}{2}-s)}{\sqrt{\pi}\Gamma(m+1-s)}\frac{x^{m-s}}{(x-x^{-1})^m}F\bigl(\tfrac{1}{2}-m, -m+s; \tfrac{1}{2}+s; x^{-2}\bigr).
\end{aligned}$$

Using the hypergeometric series (A.1) and the functional equation

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

we get

$$\begin{aligned}
\sqrt{\pi}(x-x^{-1})^m P_{s-1}^{-m}(u) &= \frac{\Gamma(-\frac{1}{2}+s)}{\Gamma(m+s)} x^{m-1+s} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-m)_n(-m+1-s)_n}{(\frac{3}{2}-s)_n} \frac{x^{-2n}}{n!} \\
&\quad + \frac{\Gamma(\frac{1}{2}-s)}{\Gamma(m+1-s)} x^{m-s} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-m)_n(-m+s)_n}{(\frac{1}{2}+s)_n} \frac{x^{-2n}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-m)_n}{n!} \biggl\{ \frac{\Gamma(-\frac{1}{2}+s)}{\Gamma(m+s)} \frac{\Gamma(\frac{3}{2}-s)\Gamma(n-m+1-s)}{\Gamma(n+\frac{3}{2}-s)\Gamma(-m+1-s)} x^{m-1+s-2n} \\
&\qquad + \frac{\Gamma(\frac{1}{2}-s)}{\Gamma(m+1-s)} \frac{\Gamma(\frac{1}{2}+s)\Gamma(n-m+s)}{\Gamma(n+\frac{1}{2}+s)\Gamma(-m+s)} x^{m-s-2n} \biggr\} \\
&= \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-m)_n}{n!} \biggl\{ \frac{\sin \pi(m+s)}{\sin \pi(-\frac{1}{2}+s)} \frac{\Gamma(n-m+1-s)}{\Gamma(n+\frac{3}{2}-s)} x^{m-1+s-2n} \\
&\qquad + \frac{\sin \pi(m+1-s)}{\sin \pi(\frac{1}{2}-s)} \frac{\Gamma(n-m+s)}{\Gamma(n+\frac{1}{2}+s)} x^{m-s-2n} \biggr\}.
\end{aligned}$$

Basic trigonometric manipulations simplify this to

$$\begin{aligned}
P_{s-1}^{-m}(u) = \frac{(-1)^m \tan(\pi s)}{\sqrt{\pi}(x-x^{-1})^m} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-m)_n}{n!} \biggl\{ &\frac{\Gamma(n-m+s)}{\Gamma(n+\frac{1}{2}+s)} x^{m-s-2n} \\
&- \frac{\Gamma(n-m+1-s)}{\Gamma(n+\frac{3}{2}-s)} x^{m-1+s-2n} \biggr\}.
\end{aligned} \tag{A.4}$$

On the right-hand side, the pole of $\tan(\pi s)$ at $s = 1/2$ is cancelled by a corresponding zero of the function defined by the sum.

For fixed $u > 1$, we consider the holomorphic function

$$\begin{aligned}
H(s) = \bigl(s(1-s)\bigr)^{(2m+1)/4} \tan(\pi s) \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-m)_n}{n!} \biggl\{ &\frac{\Gamma(n-m+s)}{\Gamma(n+\frac{1}{2}+s)} x^{m-s-2n} \\
&- \frac{\Gamma(n-m+1-s)}{\Gamma(n+\frac{3}{2}-s)} x^{m-1+s-2n} \biggr\}
\end{aligned} \tag{A.5}$$

on S_σ, where we have fixed a branch of $s \mapsto (s(1-s))^{(2m+1)/4}$. Because $H(s) = H(1-s)$, the Phragmén–Lindelöf principle gives

$$\sup_{s \in S_\sigma} |H(s)| \le \sup_{y \in \mathbf{R}} |H(\sigma+iy)|.$$

Together with (A.4), this implies

$$|P_{s-1}^{-m}(u)| \le \frac{|s(1-s)|^{-(2m+1)/4} \sup_{y \in \mathbf{R}} |H(\sigma+iy)|}{\sqrt{\pi}(4(u^2-1))^{m/2}} \quad \text{for all } s \in S_\sigma.$$

Let $y \in \mathbf{R}$ and $s = \sigma + iy$. Then we have

$$\bigl|s(1-s)\bigr|^{(2m+1)/4} = (\sigma^2+y^2)^{(2m+1)/8}((1-\sigma)^2+y^2)^{(2m+1)/8}.$$

A straightforward calculation gives

$$|\tan \pi s| = |\tan \pi(\sigma+iy)| \le \max\{1, \tan \pi\sigma\}.$$

Using Corollary A.3 and the assumption that m is even, we bound the quotients of Γ-functions appearing on the right-hand side of (A.5) independently of n:

$$\begin{aligned}\left|\frac{\Gamma(n-m+s)}{\Gamma(n+\frac{1}{2}+s)}\right| &= \frac{|\Gamma(n+s)|}{|\Gamma(n+\frac{1}{2}+s)|}\prod_{j=n-m}^{n-1}\frac{1}{|j+s|} \\ &\le \exp\left(\frac{1}{2}+\frac{1}{24(n+\sigma)(n+\frac{1}{2}+\sigma)}\right)|n+\sigma+iy|^{-1/2}\frac{1}{|\sigma+iy|^{m/2}|1-\sigma+iy|^{m/2}} \\ &\le \exp\left(\frac{1}{2}+\frac{1}{24\sigma(\frac{1}{2}+\sigma)}\right)\frac{1}{(\sigma^2+y^2)^{(m+1)/4}((1-\sigma)^2+y^2)^{m/4}}.\end{aligned}$$

This implies

$$\begin{aligned}\big|s(1-s)\big|^{(2m+1)/4}\left|\frac{\Gamma(n-m+s)}{\Gamma(n+\frac{1}{2}+s)}\right| &\le \exp\left(\frac{1}{2}+\frac{1}{24\sigma(\frac{1}{2}+\sigma)}\right) \\ &\quad\cdot\frac{(\sigma^2+y^2)^{(2m+1)/8}((1-\sigma)^2+y^2)^{(2m+1)/8}}{(\sigma^2+y^2)^{(m+1)/4}((1-\sigma)^2+y^2)^{m/4}} \\ &= \exp\left(\frac{1}{2}+\frac{1}{24\sigma(\frac{1}{2}+\sigma)}\right)\frac{((1-\sigma)^2+y^2)^{1/8}}{(\sigma^2+y^2)^{1/8}} \\ &\le \exp\left(\frac{1}{2}+\frac{1}{24\sigma(\frac{1}{2}+\sigma)}\right)\frac{(1-\sigma)^{1/4}}{\sigma^{1/4}}\end{aligned}$$

and hence

$$\big|s(1-s)\big|^{(2m+1)/4}|\tan\pi s|\left|\frac{\Gamma(n-m+s)}{\Gamma(n+\frac{1}{2}+s)}\right|\le C_\sigma.$$

Similarly,

$$\left|\frac{\Gamma(n-m+1-s)}{\Gamma(n+\frac{3}{2}-s)}\right|\le\exp\left(\frac{1}{2}+\frac{1}{24(1-\sigma)(\frac{3}{2}-\sigma)}\right)\frac{1}{((1-\sigma)^2+y^2)^{(m+1)/4}(\sigma^2+y^2)^{m/4}}$$

and

$$\begin{aligned}\big|s(1-s)\big|^{(2m+1)/4}\left|\frac{\Gamma(n-m+1-s)}{\Gamma(n+\frac{3}{2}-s)}\right| &\le \exp\left(\frac{1}{2}+\frac{1}{24(1-\sigma)(\frac{3}{2}-\sigma)}\right) \\ &\quad\cdot\frac{(\sigma^2+y^2)^{(2m+1)/8}((1-\sigma)^2+y^2)^{(2m+1)/8}}{((1-\sigma)^2+y^2)^{m/4}(\sigma^2+y^2)^{(m+1)/4}} \\ &= \exp\left(\frac{1}{2}+\frac{1}{24(1-\sigma)(\frac{3}{2}-\sigma)}\right)\frac{((1-\sigma)^2+y^2)^{1/8}}{(\sigma^2+y^2)^{1/8}} \\ &\le \exp\left(\frac{1}{2}+\frac{1}{24(1-\sigma)(\frac{3}{2}-\sigma)}\right)\frac{(1-\sigma)^{1/4}}{\sigma^{1/4}}.\end{aligned}$$

This implies

$$\big|s(1-s)\big|^{(2m+1)/4}|\tan\pi s|\left|\frac{\Gamma(n-m+1-s)}{\Gamma(n+\frac{3}{2}-s)}\right|\le C'_\sigma.$$

We conclude that

$$\begin{aligned}\sup_{y\in\mathbf{R}}|H(\sigma+iy)| &\le \sum_{n=0}^{\infty}\frac{|(\frac{1}{2}-m)_n|}{n!}\big(C_\sigma x^{m-\sigma-2n}+C'_\sigma x^{m-1+\sigma-2n}\big) \\ &= \big(C_\sigma x^{m-\sigma}+C'_\sigma x^{m-1+\sigma}\big)\sum_{n=0}^{\infty}\frac{|(\frac{1}{2}-m)_n|}{n!}x^{-2n}.\end{aligned}$$

This finishes the proof. □

With C_σ and C'_σ as in (A.3), we define an elementary function $p_\sigma\colon [1,\infty) \to \mathbf{R}$ by

$$p_\sigma(u) = \frac{C_\sigma x^{2-\sigma} + C'_\sigma x^{1+\sigma}}{4\sqrt{\pi}}\bigl((1-x^{-2})^{3/2} + 3x^{-2}\bigr), \quad \text{where } x = u + \sqrt{u^2-1}. \tag{A.6}$$

Corollary A.6. *For all $\sigma \in (0,1/2)$, $s \in S_\sigma$ and $u > 1$, we have*

$$|P^{-2}_{s-1}(u)| \le |s(1-s)|^{-5/4}\frac{p_\sigma(u)}{u^2-1}.$$

Proof. We note that

$$|(-\tfrac{3}{2})_n| = \begin{cases} -(-\frac{3}{2})_n & \text{for } n = 1, \\ (-\frac{3}{2})_n & \text{otherwise.} \end{cases}$$

This implies that for $z \in (0,1)$, we have

$$\begin{aligned} \sum_{n=0}^\infty \frac{|(-\frac{3}{2})_n|}{n!} z^n &= \sum_{n=0}^\infty \frac{(-\frac{3}{2})_n}{n!} z^n - 2\frac{(-\frac{3}{2})_1}{1!} z^1 \\ &= (1-z)^{3/2} + 3z. \end{aligned}$$

The claim immediately follows from this identity and Proposition A.5. □

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